Randomness via effective descriptive set theory

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Limits on computability, definability, provability
Celebrating the mathematical and professional contributions of
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My interactions with CT, the IMS, and maths at NUS

- IMS programmes and workshops, starting with "Computational Prospects of Infinity" 2005.
  Talks, and papers in the corresponding proceedings volumes.

- Visits, about yearly

- NUS logic seminar talks

- 2008 joint paper with CT and Liang on higher randomness

Idea of randomness via definable tests

- We study randomness of subsets of ω, and of reals in $[0, 1]$.
- Intuitively, random should mean "typical". An object x is random if it is in no null set.

Γ-randomness

- Choose a definability restriction Γ on null sets.
- One says that x is Γ-random if it is in no Γ null set.

Null sets satisfying Γ are called Γ-tests. As there are only countably many tests, random objects in this sense exist.

1970-1975:

The dawn of

randomness via

effective descriptive set theory

Two short papers of Per Martin-Löf : 1966/1970

[1966] The definition of random sequences.
Information and Control 9. (1666 citations on GS)

- Defines tests as sequences of uniformly Σ^0_1 sets with recursively bounded measure going to 0.
- Proves existence of universal test.

[1970] On the notion of randomness. In Proceedings of a 1968 conference on Intuitionism and Proof Theory. (88 citations on GS)

- Defines tests as Δ^1_1 null sets.
- Shows the collection of Δ^1_1 randoms is Σ^1_1.
- It is not Π^1_1 by Sacks-Tanaka.
- In particular, there is no universal test.

1970: Solovay's Theorem and random reals

Assuming there exists an inaccessible cardinal, Solovay built a model of ZF+ DC in which every set of reals is Lebesgue measurable.

From the paper:

> II. THE CONCEPT OF A RANDOM REAL
>
> We first discuss, in II.1, the relation between Borel sets of a countable transitive model $\mathfrak{M}$ and Borel sets of the real world. This is a preliminary to a study of the key concept of this paper, the concept of a random real. This is our main tool in adapting Cohen's method to measure theoretic problems.

- Solovay used a forcing where the conditions are Borel sets of positive measure coded in the ground model M.
- An M-generic ultrafilter G corresponds to a random real x_G over M (Jech, Set theory, 2002 Lemma 26.2).

1975: Kechris and Stern papers

- Mycielski and Swierczkowski (1964):
  Under projective determinacy (PD), each projective set of reals is Lebesgue measurable.
- Kechris (1975): There is a largest Π^1_1 null set. Under PD, for each $n \geq 1$ there is a largest Π^1_{2n+1} and a largest Σ^1_{2n} null set.
- Stern (1975):
  - the largest Π^1_1 null set equals the union all the Δ^1_1 null sets and $\{A \colon \omega_1^A > \omega_1^{\mathrm{CK}}\}$ (that is, some A-computable well-oder is noncomputable; equivalently, $\mathcal{O} \leq_h A$).
  - If almost all reals are random over L, then the largest Σ^1_2 null set equals the set of non-random reals over L.

Similar results hold for category.

Towards understanding algorithmic randomess

- Solovay worked with Chaitin in 1975 on the algorithmic theory of randomness, leading to a now famous set of notes.
- Until 2000, the development of algorithmic randomness proceeded slowly but steadily, with work by researchers with computability background such as Calude, Demuth, Kučera, Muchnik, Shen, Terwijn, Zambella and others.
- By the early 2000s it was generally accepted that randomness helps to understand computational complexity.
  - Terwijn & Kučera(1999) constructed an r.e., non-computable oracle that is low for ML-randomness: each MLR is MLR relative to A.
  - N. (2002) showed that K-trivial = low for ML-randomness, building on work with Downey, Hirschfeldt, and Stephan.
- Textbooks leading to research level:
  Computability and Randomness (N., 2009)

2007 - 2015

Rediscovery as “Higher randomness”

and development

Π^1_1 as an existential property

Recall that $A \subseteq \omega$ is Π^1_1 iff it is of the form $\{n \colon \forall X\ R(X,n)\}$ where R is arithmetical.

Spector-Gandy

- $A \subseteq \omega$ is $\Pi^1_1 \iff$ there is a Σ_1 formula $\phi(n)$ such that
$$A = \{n \in \omega \colon L(\omega_1^{CK}) \models \phi(n)\}.$$
- $\mathcal{S} \subseteq \mathcal{P}(\omega)$ is $\Pi^1_1 \iff$ there is a Σ_1 formula $\phi(X)$ such that
$$\mathcal{S} = \{X \subseteq \omega \colon L(\omega_1^{X})[X] \models \phi(X)\}.$$

We can think of the existential quantifier in the Σ_1 formulas to be over recursive ordinals (resp, over X-recursive ordinals).
So, a Π^1_1 set can be thought of as enumerated along recursive ordinal stages.

Hjorth and N. paper

- 2007 Hjorth and N. published a J. London Math Soc. paper "Randomness via effective descriptive set theory". Ted Slaman had suggested to me to study this.

- Idea: develop higher randomness, based on Δ^1_1 and Π^1_1 tests, in place of computable/r.e. tests. Use the idea of enumeration along recursive ordinals in Spector-Gandy.

- For instance, a Π^1_1 ML-test is a uniformly Π^1_1 sequence of open sets $(U_n)_{n\in\mathbb{N}}$ such that $\lambda U_n \leq 2^{-n}$.

- Π^1_1 randomness has no analogue in the algorithmic setting.

Strict implications

Π^1_1 randomness $\Rightarrow$ Π^1_1 ML-randomness $\Rightarrow$
Δ^1_1 randomness $=$ Δ^1_1 ML-randomness.

- Hjorth and N. introduced a Π^1_1 version of the theory of prefix free machines, and thereby higher prefix-free Kolmogorov complexity and higher Ω, denoted $\underline{K}$ and $\underline{\Omega}$.
- They proved a version of Schnorr-Levin, showed $\underline{\Omega}$ is Π^1_1 ML-random, and $\underline{\Omega} \equiv_T O$. Thus $\underline{\Omega}$ is not Π^1_1 random.
- For the second separation, they built a Δ^1_1 random of slowly growing initial segment complexity.

Recall Stern's result: the largest Π^1_1 null set equals the union of all Δ^1_1 null sets and $\{A\colon \omega_1^A > \omega_1^{\mathrm{CK}}\}$. Thus, the three randomness notions can only differ for sets A such that $\omega_1^A > \omega_1^{\mathrm{CK}}$.

Lowness for Π^1_1 ML-randomness

- Allowing test defining a randomness notion $\mathcal{R}$ access to an oracle A generally strengthens them: $\mathcal{R}^A$ is a proper subset of $\mathcal{R}$.
  An oracle A is low for $\mathcal{R}$ if NOT: $\mathcal{R}^A = \mathcal{R}$.
- Often this is an interesting class. E.g., low for ML$=$ K-trivial.

Theorem (Hjorth and N., 2007)

If A is low for Π^1_1 ML-randomness then A is hyperarithmetic.

- The proof uses the methods of the algorithmic setting to show A is higher K-trivial.
- Also a higher K-trivial that is not hyperarithmetic is $\geq_H \mathcal{O}$. This can't be for our set A, so A is hyperarithmetic.

Chong, N., Yu 2008: Hyp-domination

Set theory: If x is M-random for a model M of ZF+DC, then every function $\omega \to \omega$ in $M[x]$ is dominated by a function in M.

- Chong, N., Yu, IJM 2008: "Lowness of higher randomness notions" called $A \subseteq \omega$ hyp-dominated if each function hyperarithmetic in A is dominated by a hyperarithmetic function.
- They showed that each Π^1_1 random A is hyp-dominated.
- This contrasts with the case of algorithmic randomness: each ML-random relative to the halting problem (or even, each Demuth random) is of hyper-immune degree.

Hyp-domination and characterising Π^1_1 randomness

Recall that each Π^1_1 random A is hyp-dominated: each function hyperarithmetic in A is dominated by a hyperarithmetic function.

Kjos-Hanssen, N., Stephan and Yu, 2009

A is Π^1_1 random $\Longleftrightarrow$
A is hyp-dominated and in no closed Π^1_1 null class.

Thus, inside the class of hyp-dominated sets, everything from Π^1_1 Kurtz to full Π^1_1 randomness collapses.

More on Δ^1_1 randomness

- Recall $Z \subseteq \omega$ is Δ^1_1 random (Martin-Löf, Stern) if Z is in no Δ^1_1 null set.

- This is the higher analog of both computable randomness (martingales) and Schnorr randomness (test components have uniformly computable measures).

- By the methods of Brattka, Miller and N. 2016,
  a real $r \in [0, 1]$ is Δ^1_1 random if and only if each hyperarithmetic monotonic function $g\colon [0, 1] \to \mathbb{R}$ is differentiable at r.

- The same holds for hyperarithmetic functions of bounded variation. This contrasts with the algorithmic case: the notion

Lowness for Δ^1_1 randomness

- A hyperarithmetic trace for $f \in {}^\omega\omega$ is a hyperarithmetic sequence of finite sets $(L_n)_{n\in\mathbb{N}}$ such that $\forall n[f(n) \in L_n]$.
- The following is an analog of a result for Schnorr randomness by Terwijn and Zambella.

Theorem (Chong, N., Yu, 2009)

A is low for Δ^1_1 randomness $\iff$ there is a hyperarithmetic bound $h \in {}^\omega\omega$ such that each $f \leq_h A$ has a hyperarithmetic trace $(L_n)_{n\in\mathbb{N}}$ with $|L_n| \leq h(n)$.

Deeper understanding of Π^1_1 randomness

Chong, Yu; Greenberg, Monin

until 2021

So, what about Π^1_1 randomness?

The definition of Π^1_1 randomness is alluringly simple, because there is a universal test. What do we know?

Chong and Yu: analog of Demuth's theorem.

The h-degrees of Π^1_1 randoms are closed downwards.

Benoit Monin's 2014 thesis, with involvement of Greenberg/ Greenberg and Monin, 2017

- Π^1_1 randomness is Borel, in fact $\mathbf{\Pi}^0_3$, and this is optimal.
- If an oracle low for Π^1_1-randomness, it is hyperarithmetic.

Genericity in (higher) computability

- Ultrafilters G are oracles (subsets of ω)
- Algorithmic/ definability restriction on dense sets.

Cohen forcing: $\mathbb{P} = 2^{<\omega}$.
G is 1-generic if for each effective list $[\sigma_n]_{n\in\mathbb{N}}$ of cones, either $G \in \bigcup_n [\sigma_n]$, or $\exists [\tau]$ with $[\tau] \cap \bigcup_n [\sigma_n] = \emptyset$ such that $G \in [\tau]$.

Random forcing: $\mathbb{P} = \Sigma^1_1$ closed sets of positive measure with inclusion.
G is Σ^1_1-Solovay generic (Monin) if for each effective list $(\mathcal{F}_n)_{n\in\mathbb{N}}$ in $\mathbb{P}$, either $G \in \bigcup_n \mathcal{F}_n$, or $\exists \mathcal{L} \in \mathbb{P}$ with $\mathcal{L} \cap \bigcup \mathcal{F}_n = \emptyset$ such that $G \in \mathcal{L}$.

Characterise Π^1_1 random by genericity

Recall :

- $\mathbb{P} = \Sigma^1_1$ closed sets of positive measure.
- G is Σ^1_1-Solovay generic if for each effective list $(\mathcal{F}_n)_{n\in\mathbb{N}}$ in $\mathbb{P}$, either $G \in \bigcup_n \mathcal{F}_n$, or $\exists \mathcal{L} \in \mathbb{P}$ with $\mathcal{L} \cap \bigcup \mathcal{F}_n = \emptyset$ such that $G \in \mathcal{L}$.

Theorem (Monin, 2014)

$$Z \text{ is } \Pi^1_1 \text{ random} \iff Z \text{ is } \Sigma^1_1 \text{ Solovay generic.}$$

Can check that being Σ^1_1 Solovay generic is $\mathbf{\Pi}^0_3$.
The hard bit: if Z is Σ^1_1 Solovay generic then $\omega_1^Z = \omega_1^{CK}$.

Lowness for Π^1_1 randomness

The characterisation in terms of Solovay genericity, together with Hjorth/N. result, was used to show

A is non-hyperarithmetic $\Rightarrow$ some Π^1_1 random is not $\Pi^1_1(A)$ ML random

In particular it is not $\Pi^1_1(A)$ random.

Simplified proofs in the final chapter of Chong & Yu's book (2015).

fin-h and higher Turing reducibilitiy

Because of the infinite computation time, an unrestricted computation can look at the whole oracle.
To get results from the mother lode of algorithmic randomness into the higher setting we need more restricted reductions.

- fin-h (Hjorth and N.): functional given as Π^1_1 set of conditions σ, τ, which is consistent (monotonic)
  Proved version of Kučera-Gacs.
- hT (Bienvenu, Greenberg, Monin): discards consistency.
  Convincing reasons that this is better.
- See their papers: Continuous higher randomness (JML 1017), Bad oracles (2021)

Book references on higher randomness

Final chapters in three books

- Computability and randomness (N., 2009)
- Recursion theory (Chong and Yu, 2015)
- Computability Theory Algorithmic Randomness, Reverse Mathematics and Higher Computability Theory (Monin and Patey, PiL, to appear)