Superhighness and strong jump traceability

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Abstract. Let A be a c.e. set. Then A is strongly jump traceable if and only if A is Turing below each superhigh Martin-Löf random set. The proof combines priority with measure theoretic arguments.

1 Introduction

A lowness property of a set $A \subseteq \mathbb{N}$ specifies a sense in which A is computationally weak.

(I) Usually this means that A has limited strength when used as an oracle. An example is superlowness, $A' \le_{\mathrm{tt}} \emptyset'$. Further examples are given by traceability properties of A. Such a property specifies how to effectively approximate the values of certain functions (partial) computable in A. For instance, A is *jump traceable* [1] if $J^A(n)\downarrow$ implies $J^A(n) \in T_n$, for some uniformly c.e. sequence $(T_n)_{n\in\mathbb{N}}$ of computably bounded size. Here J is the jump functional: If $X \subseteq \mathbb{N}$, we write $J^X(n)$ for $\Phi_n^X(n)$.

(II) A further way to be computationally weak is to be easy to compute. A lowness property of this kind specifies a sense in which many oracles compute A. For instance, consider the property to be a base for ML-randomness, introduced in [2]. Here the class of oracles computing A is large enough to admit a set that is ML-random relative to A. By [3] this property coincides with the type (I) lowness property of being low for ML-randomness.

As our main result, we show a surprising further coincidence of a type (I) and a type (II) lowness property for c.e. sets. The type (I) property is strong jump traceability, introduced in [4], and studied in more depth in [5]. We say that a computable function $h\colon \mathbb{N} \to \mathbb{N} \setminus \{0\}$ is an *order function* if h is nondecreasing and unbounded.

Definition 1. *$A \subseteq \mathbb{N}$ is* strongly jump traceable *(s.j.t.) if for each order function h, there is a uniformly c.e. sequence $(T_n)_{n\in\mathbb{N}}$ such that $\forall n\, |T_n| \le h(n)$ and $\forall n\, [J^A(n)\downarrow \to\ J^A(n) \in T_n]$.*

Figueira, Nies and Stephan [4] built a promptly simple set that is strongly jump traceable. Cholak, Downey and Greenberg [5] showed that the strongly jump traceable c.e. sets form a proper subideal of the K-trivial c.e. sets under Turing reducibility.

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We say that a set $Y \subseteq \mathbb{N}$ is *superhigh* if $\emptyset'' \leq_{\mathrm{tt}} Y'$. This notion was first studied by Mohrherr [6] for c.e. sets. For background and results on superhighness see [7, 8]. The type (II) property is to be Turing below each superhigh ML-random set. Thus our main result is that *a c.e. set A is strongly jump traceable* if and only if *A is Turing below each superhigh Martin-Löf random set.*

The property to be Turing below each superhigh ML-random set can be put into a more general context. For a class $\mathcal{H} \subseteq 2^\omega$, we define the corresponding diamond class

$$\mathcal{H}^\diamond = \{A\colon\ A \text{ is c.e. } \& \ \forall Y \in \mathcal{H} \cap \mathsf{MLR}\, [A \leq_T Y]\}.$$

Here MLR is the class of ML-random sets. Note that $\mathcal{H}^\diamond$ determines an ideal in the c.e. Turing degrees. By a result of Hirschfeldt and Miller (see [7, 5.3.15]), for each null Σ^0_3 class, the corresponding diamond class contains a promptly simple set A. Their construction of A is via a non-adaptive cost-function construction (see [7, Section 5.3] for details on cost functions). That is, the cost function can be given in advance. This means that the construction can be viewed as injury-free. In contrast, the direct construction of a promptly simple strongly jump traceable set in [4] varies Post's construction of a low simple set, and therefore has injury.

In [9] a result similar to our main result was obtained when $\mathcal{H}$ is the class of superlow sets Y (namely, $Y' \leq_{\mathrm{tt}} \emptyset'$). Earlier, Hirschfeldt and Nies had obtained such a coincidence for the class $\mathcal{H}$ of ω-c.e. sets Y (namely, $Y' \leq_{\mathrm{tt}} \emptyset'$).

In all cases, to show that a c.e. strongly jump traceable set A is in the required diamond class, one finds an appropriate collection of benign cost functions; this key concept was introduced by Greenberg and Nies [10]. The set A obeys each benign cost function by the main result of [10]. This implies that A is in the diamond class.

It is harder to prove the converse inclusion: each c.e. set in $\mathcal{H}^\diamond$ is s.j.t. Suppose an order function h is given. For one thing, similar to the proof of the analogous inclusion in [9], we use a variant of the golden run method introduced in [12]. One wants to restrict the changes of A to the extent that A is strongly jump traceable. To this end, one attempts to define a "naughty set" $Y \in \mathcal{H} \cap \mathsf{MLR}$. It exploits the changes of A in order to avoid being Turing above A. The number of levels in the golden run construction is infinite, with the e-th level based on the Turing functional Φ_e. If the golden run fails to exist at level e then $A \neq \Phi_e^Y$. If this is so for all e then $A \not\leq_T Y$, contrary to the hypothesis that $A \in \mathcal{H}^\diamond$. Hence a golden run must exist. Since it is golden it successfully builds the required trace for J^A with bound h.

A further ingredient of our proof stems from ideas that started in Kurtz [13] and were elaborated further, for instance, in Nies [12, 14]: mixing priority arguments and measure theoretic arguments. In contrast, the proof in [9] is not measure theoretic. (Indeed, they prove, more generally, that for *each* non-empty Π^0_1 class P, each c.e. set Turing below every superlow member of P must be strongly jump traceable. This stronger statement has no analog for superhighness, for instance because all members of P could be computable.)

Here we need to make the naughty set Y superhigh. This is done by coding $\emptyset''$ (see [7, 3.3.2]) in the style of Kučera, but not quite into Y: the coding strings change due to the activity of the tracing procedures. The number of times they change is computably bounded. So the coding yields $\emptyset'' \leq_{\mathrm{tt}} Y'$.

Notation. Suppose f is a unary function and $\widetilde{f}$ is binary. We write

$$\forall n\, f(n) = \lim_s^{\mathrm{comp}} \widetilde{f}(n,s)$$

if there is a computable function $g : \mathbb{N} \to \mathbb{N}$ such that for all n, the set

$$\{s > 0\colon\ \widetilde{f}(n,s) \neq \widetilde{f}(n,s-1)\}$$

has cardinality less than $g(n)$, and $\lim_s \widetilde{f}(n,s) = f(n)$.

We let $X' = \{n\colon\ J^X(n)\downarrow\}$, and $X'_t = \{n\colon\ J^X_t(n)\downarrow\}$. We use Knuth's bracket notation in sums. For instance, $\sum_n n^{-2} [\![n \text{ is odd}]\!]$ denotes $1 + 1/9 + 1/25 + \ldots = \pi^2/8$.

A forthcoming paper by Greenberg, Hirschfeldt and Nies (Characterizing the s.j.t. sets via randomness) contains a new proof of Theorem 2 using the language of "golden pairs". This makes it possible to cut some parameters.

2 Benign cost functions and $\mathsf{Shigh}^{\diamond}$

Note that a function f is d.n.c. relative to $\emptyset'$ if $\forall x\, \neg f(x) = J^{\emptyset'}(x)$. Let P be the $\Pi^0_1(\emptyset')$ class of $\{0,1\}$-valued functions that are d.n.c. relative to $\emptyset'$. The PA sets form a null class (see, for instance, [7, 8.5.12]). Relativizing this to $\emptyset'$, we obtain that the class $\{Z\colon\ \exists f \leq_T Z \oplus \emptyset' [f \in P]\}$ is null. Then, since $\mathrm{GL}_1 = \{Z\colon\ Z' \equiv_T Z \oplus \emptyset'\}$ is conull, the following class, suggested by Simpson, is also null:

$$\mathcal{H} = \{Z\colon\ \exists f \leq_{\mathrm{tt}} Z' [f \in P]\}. \tag{1}$$

This class clearly contains Shigh because $\emptyset''$ truth-table computes a function that is d.n.c. relative to $\emptyset'$. Since $\mathcal{H}$ is Σ^0_3, by a result of Hirschfeldt and Miller (see [7, 5.3.15]) the class $\mathcal{H}^{\diamond}$ contains a promptly simple set. We strengthen this:

Theorem 1. *Let A be a c.e. set that is strongly jump traceable. Then $A \in \mathcal{H}^{\diamond}$.*

Proof. In [10] a cost function c is defined to be *benign* if there is a computable function g with the following property: if $x_0 < \ldots < x_n$ and $c(x_i, x_{i+1}) \geq 2^{-e}$ for each i, then $n \leq g(e)$. For each truth table reduction Γ we define a benign cost function c such that for each Δ^0_2 set A, and each ML-random set Y,

$$A \text{ obeys } c \text{ and } \Gamma^{Y'} \text{ is } \{0,1\}\text{-valued d.n.c. relative to } \emptyset' \Rightarrow A \leq_T Y.$$

Let (I_e) be the sequence of consecutive intervals of $\mathbb{N}$ of length e. Thus $\min I_e = e(e+1)/2$. We define a function $\alpha \leq_T \emptyset'$. We are given a partial computable function p and (via the Recursion Theorem) think of p as a reduction function for α, namely, p is total, increasing, and $\forall x\ \alpha(x) \simeq J^{\emptyset'}(p(x))$.

At stage s of the construction we define the approximation $\alpha_s(x)$. Suppose $x \in I_e$. If $p(y)$ is undefined at stage s for some $y \in I_e$ let $\alpha_s(x) = 0$. Otherwise, let

$$\mathcal{C}_{e,s} = \{Y\colon\ \exists t_{\,v \le t \le s} \forall x \in I_e\,[1 - \alpha_t(x) = \Gamma(Y'_t, p(x))]\}, \qquad (2)$$

where $v \le s$ is greatest such that $v = 0$ or $\alpha_v \restriction I_e \neq \alpha_{v-1} \restriction I_e$. (Thus, $\mathcal{C}_{e,s}$ is the set of oracles Y such that Y' computes α correctly at some stage t after the last change of $\alpha \restriction_{I_e}$.)

Construction of α.

Stage $s > 0$. For each $e < s$, if $\lambda\mathcal{C}_{e,s-1} \le 2^{-e+1}$ let $\alpha_s \restriction I_e = \alpha_{s-1} \restriction I_e$. Otherwise change $\alpha \restriction I_e$: define $\alpha_s \restriction I_e$ in such a way that $\lambda\mathcal{C}_{e,s} \le 2^{-e}$.

Claim. *$\alpha(x) = \lim_s \alpha_s(x)$ exists for each x.*

We use a measure theoretic fact suggested by Hirschfeldt in a related context (see [7, 1.9.15]). Suppose $N, e \in \mathbb{N}$, and for $1 \le i \le N$, the class $\mathcal{B}_i$ is measurable and $\lambda\mathcal{B}_i \ge 2^{-e}$. If $N > k2^e$ then there is a set $F \subseteq \{1, \ldots, N\}$ such that $|F| = k+1$ and $\bigcap_{i \in F} \mathcal{B}_i \neq \emptyset$.

Suppose now that $0 = v_0 < v_1 < \ldots < v_N$ are consecutive stages at which $\alpha \restriction I_e$ changes. Thus $p \restriction I_e$ is defined. Then $\lambda\mathcal{B}_i \ge 2^{-e}$ for each $i \le N$, where

$$\mathcal{B}_i = \{Y\colon\ Y'_{v_{i+1}} \restriction_k \neq Y'_{v_i} \restriction_k\},$$

and $k = \mathsf{use}\ \Gamma(\max p(I_e))$, because $\lambda\mathcal{C}_e$ increased by at least 2^{-e} from v_i to v_{i+1}. Note that the intersection of any $k+1$ of the $\mathcal{B}_i$ is empty. Thus $N \le 2^e k$ by the measure theoretic fact. $\diamond$

Since α is Δ^0_2, by the Recursion Theorem, we can now assume that p is a reduction function for α. Then in fact we have a computable bound g on the number of changes of $\alpha \restriction I_e$ given by $g(e) = 2^e \mathsf{use}\ \Gamma(\max p(I_e))$.

To complete the proof, let A be a c.e. set that is strongly jump traceable. We define a cost function c by $c(x, s) = 2^{-x}$ for each $x \ge s$; if $x < s$, and $e \le x$ is least such that $e = x$ or $\alpha_s \restriction I_e \neq \alpha_{s-1} \restriction I_e$, let

$$c(x, s) = \max(c(x, s-1), 2^{-e}).$$

Note that the cost function c is benign as defined in [10]: if $x_0 < \ldots < x_n$ and $c(x_i, x_{i+1}) \ge 2^{-e}$ for each i, then $\alpha_s \restriction I_e \neq \alpha_{s-1} \restriction I_e$ for some s such that $x_i < s \le x_{i+1}$. Hence $n \le g(e)$ where g is defined after the claim.

By [10] fix a computable enumeration $(A_s)_{s\in\mathbb{N}}$ of A that obeys c. (The rest of the argument actually works for a computable approximation $(A_s)_{s\in\mathbb{N}}$ of a Δ^0_2 set A.)

We build a Solovay test $\mathcal{G}$ as follows: when $A_{t-1}(x) \neq A_t(x)$, we put $\mathcal{C}_{e,t}$ defined in (2) into $\mathcal{G}$ where e is largest such that $\alpha \restriction I_e$ has been stable from x to t. Then $2^{-e} \le c(x, t)$. Since $\lambda\mathcal{C}_{e,t} \le 2^{-e+1} \le 2c(x, t)$ and the computable approximation of A obeys c, $\mathcal{G}$ is indeed a Solovay test.

Choose s_0 such that $\sigma \not\preceq Y$ for each $[\sigma]$ enumerated into $\mathcal{G}$ after stage s_0. To show $A \le_T Y$, given an input $y \ge s_0$, using Y as an oracle, compute $s > y$ such that $\alpha_s(x) = \Gamma(Y'_s; x)$ for each $x < y$. Then $A_s(y) = A(y)$: if $A_u(y) \neq A_{u-1}(y)$ for $u > s$, let $e \le y$ be largest such that $\alpha \restriction I_e$ has been stable from y to u.

Then by stage $s > y$ the set Y is in $\mathcal{C}_{e,s} \subseteq \mathcal{C}_{e,t}$, so we put Y into $\mathcal{G}$ at stage u, contradiction.

In the following we give a direct construction of a null Σ^0_3 class containing the superhigh sets. Note that the class $\mathcal{H}$ defined in (1) is such a class. However, the proof below uses techniques of independent interest. For instance, they might be of use to resolve the open question whether superhighness itself is a Σ^0_3 property.

Proposition 1. *There is a null Σ^0_3 class containing the superhigh sets.*

Proof. For each truth-table reduction Φ, we uniformly define a null Π^0_2 class $\mathcal{S}_\Phi$ such that $\emptyset'' = \Phi(Y') \;\rightarrow\; Y \in \mathcal{S}_\Phi$.

We build a Δ^0_2 set D_Φ. Then, by the Recursion Theorem we have a truth-table reduction Γ_Φ such that $\emptyset'' = \Phi(Y') \;\rightarrow\; D_\Phi = \Gamma(Y')$. We define D_Φ in such a way that $\mathcal{S}_\Phi = \{Y\colon\, D_\Phi = \Gamma(Y')\}$ is null. Also, $\mathcal{S}_\Phi$ is Π^0_2 because

$$Y \in \mathcal{S}_\Phi \;\leftrightarrow\; \forall w\, \forall i > w \exists s > i\, D_\Phi(w, s) = \Gamma(Y'_s; w).$$

Claim. *For each string σ, the real number $r_\sigma = \lambda\{Z\colon\, \sigma \prec Z'\}$ is the difference of left-c.e. reals uniformly in σ* (see [7, 1.8.15]).
To see this, note that for each finite set F the class $\mathcal{C}_F = \{Z\colon F \subseteq Z'\}$ is uniformly Σ^0_1. Let $F(\sigma) = \{j < |\sigma|\colon\, \sigma(j) = 1\}$, then

$$r_\sigma = \lambda(\mathcal{C}_{F(\sigma)} - \textstyle\bigcup_{r<|\sigma|\&\sigma(r)=0} \mathcal{C}_{\{r\}\cup F(\sigma)}).$$

This proves the claim. Now, for each τ let $b_\tau = \lambda\{Z\colon\, \tau \prec \Gamma(Z')\}$. Then $b_\tau = \sum_\sigma r_\sigma \,[\![\tau = \Gamma^\sigma]\!]$ is uniformly difference left-c.e.

One can define the Δ^0_2 set $D = D_\Phi$ in such a way that $2b_{D\restriction n+1} \leq b_{D\restriction n}$ for each n. Then $2^{-n} \geq \lambda\{Y\colon\, D_\Phi \restriction_n = \Gamma(Y')\restriction_n\}$ for each n, so $\mathcal{S}_\Phi$ is null.

3 Each set in $\mathsf{Shigh}^\diamond$ is strongly jump traceable

Theorem 2. *Let A be a c.e. set that is Turing below all ML-random superhigh sets. Then A is strongly jump traceable.*

Proof. Let h be an order function. We will define a ML-random superhigh set Z such that $A \leq_T Z$ implies that A is jump traceable via bound h. In fact for an arbitrary given set G we can define Z such that $G \leq_{\mathrm{tt}} Z'$. If also $G \geq_{tt} \emptyset''$, then Z is superhigh.

Preliminaries. Let λ denote the uniform measure on Cantor space. We will need a lower bound on the measure of a non-empty Π^0_1 class of ML-random sets. This bound is given uniformly in an index for the class (Kučera; see [7, 3.3.3]). Let $Q_0 \subseteq \mathsf{MLR}$ be the complement $2^\omega - \mathcal{R}_1$ of the second component of the standard universal ML-test.

Lemma 1. *Given an effective listing $(P^v)_{v\in\mathbb{N}}$ of Π^0_1 classes, $P^v \subseteq Q_0$, there is a constant c_0 such that $\lambda P^v \leq 2^{-K(v)-c_0} \rightarrow P^v = \emptyset$.*

We assume an indexing of all the Π^0_1 classes. Given an index for a Π^0_1 class P we have an effective approximation $P = \bigcap_t P_t$ where P_t is a clopen set ([7, Section 1.8]).

The basic set-up. For each e, a procedure R^e (with further parameters to be discussed later) builds a c.e. trace $(T_x)_{x\in\mathbb{N}}$ with bound h. Either for almost all x, $J^A(x)\downarrow$ implies $J^A(x) \in T_x$, or R^e shows that $A \neq \Phi^Z_e$. Since Z is superhigh, the first alternative must hold for some e.

When a new computation $w = J^A(x)\downarrow$ with use u appears, R^e activates a sub-procedure S^e_x. This sub-procedure waits for evidence that $A\restriction_u$ is stable before putting w into the trace set T_x. By first waiting long enough, it makes sure that an $A\restriction_u$ change after this tracing can happen for at most $h(x)$ times, so that $|T_x| \leq h(x)$. S^e_x also calls an instance of the next procedure R^{e+1}. Thus, during the construction we can have many runs of each of the procedures R^e and S^e_x.

The environment of a procedure. Each R^e has as further parameters a Π^0_1 class P and a number $r \in \mathbb{N}$. It assumes that $Z \in P$ and $2^{-r} < \lambda P$. Each S^e_x activated by $R^e(P, r)$ will specify an appropriate subclass $Q \subseteq P$ and a number $q \in \mathbb{N}$, and call $R^{e+1}(Q, q)$.

Initially we call $R^0(Q_0, 2)$

The two phases of S^e_x. A procedure S^e_x alternates between Phases I, and II. When changing phases it returns control to R^e. In our first approximation to describing the construction, once a computation $w = J^A(x)\downarrow$ with use u appears, S^e_x enters Phase I. It considers the Σ^0_1 class $C = \{Z\colon \Phi^Z_e\restriction_u = A\restriction_u\}$. It calls $R^{e+1}(Q, q)$ where $Q = P - C$ and q is obtained by Lemma 1. If it stays here then, because $Z \in Q$, its outcome is that $\Phi^Z_e \neq A$.

For a threshold δ depending only on r and x, once $\lambda(P_s \cap C_s) > \delta$ at stage s it lets $D = C_s$ and puts w into T_x. Now the outcome is that $J^A(x)$ has been traced. So S^e_x can return and stay inactive unless $A\restriction_u$ changes.

Once $A\restriction_u$ has changed, S^e_x enters Phase II by calling $R^{e+1}(Q, q)$ where now $Q = P \cap D$ and q is obtained by Lemma 1. Its outcome is again that $\Phi^Z_e \neq A$, this time because $\Phi^Z_e\restriction_u$ is the previous value of $A\restriction_u$ (here we use that A is c.e.).

If, later on, $P \cap D$ becomes empty, then S^e_x returns. It is now turned back to the beginning and may start again in Phase I when a new computation $J^A(x)$ appears. Note that P has now lost a measure of δ. So S^e_x can go back to Phase I for at most $1/\delta$ times.

The golden run. For some e we want a run of R^e such that each sub-procedure S^e_x it calls returns. For then, the c.e. trace $(T_x)_{x\in\mathbb{N}}$ this run of R^e builds is a trace for J^A. If no such run R^e exists then each run of R^e eventually calls some S^e_x which does not return, and therefore permanently runs a procedure R^{e+1}. If $Z \in \bigcap P_e$ where P_e is the parameter of the final run of a procedure R^e, then $A \not\leq_T Z$. So we have a contradiction if we can define a set $Z \in \bigcap_e P_e$ such that $G \leq_{\mathrm{tt}} Z'$.

Ensuring that $G \leq_{\mathrm{tt}} Z'$. For this we have to introduce new parameters into the procedures S^e_x.

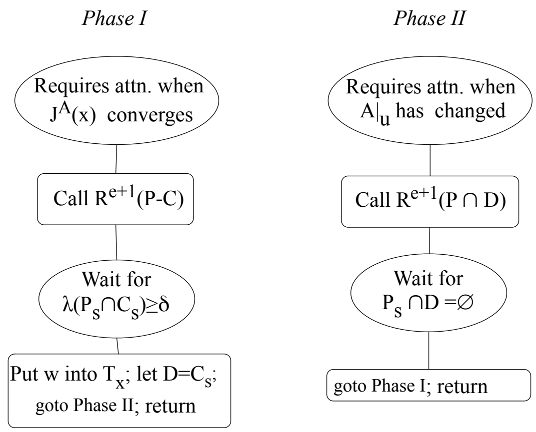

Fig. 1. Diagram for the procedure S_x^e

Note that $G \le_{tt} Z'$ iff there is a binary function $f \le_T Z$ such that $\forall x\, G\restriction_x = \lim_s^{\text{comp}} f(x,s)$ (namely, the number of changes is computably bounded). We will define Z such that Z' encodes G. We use a variant of Kučera's method to code into ML-random sets. We define strings $z_\gamma = \lim_s^{\text{comp}} z_{\gamma,s}$ and let $Z = \bigcup_{\gamma \prec G} z_\gamma$. The strings $z_{\gamma,s}$ are given effectively, and for each s they are pairwise incomparable. Then we let $f(x,s) = \gamma$ if $|\gamma| = x$ and $z_{\gamma,s} \prec Z$, and $f(x,s) = \varnothing$ if there is no such γ.

Firstly, we review Kučera's coding into a member of a Π_1^0-class P of positive measure. For a string x let $\lambda(P|x) = 2^{|x|}\lambda(P \cap [x])$.

Lemma 2 (Kučera; see [7], 3.3.1). *Suppose that P is a Π_1^0 class, x is a string, and $\lambda(P|x) \ge 2^{-l}$ where $l \in \mathbb{N}$. Then there are at least two strings $w \succeq x$ of length $|x|+l+1$ such that $\lambda(P|w) > 2^{-l-1}$. We let w_0 be the leftmost and w_1 be the rightmost such string.*

In the following we code a string β into a string y_β on a Π_1^0 class P.

Definition 2. Given a Π_1^0 class P, a string z such that $P \subseteq [z]$, and $r \in \mathbb{N}$ such that $2^{-r} < \lambda P$, we define a string

$$y_\beta = \mathsf{kuc}(P, r, z, \beta)$$

as follows: $y_\varnothing = z$; if $x = y_\beta$ has been defined, let $l = r + |\beta|$, and let $y_{\beta\frown b} = w_b$ for $b \in \{0,1\}$, where the strings w_b are defined as in Lemma 2.

Note that for each β we have $\lambda(P \mid y_\beta) \ge 2^{-r-|\beta|}$ and

$$|y_\beta| \le |z| + |\beta|(r + |\beta| + 1). \tag{3}$$

At stage s we have the approximation $y_{\beta,s} = \mathsf{kuc}(P_s \cap [z], r, z, \beta)$. While $y_{\beta,s}$ is stable, the string w_b in the recursive definition above changes at most 2^l times. Thus, inductively, $y_{\beta,s}$ changes at most $2^{|\beta|(r+|\beta|+1)}$ times.

For each e, η we may have a version of R^e denoted $R^{e,\eta}(P, r, z_\eta)$. It assumes that η has already been coded into the initial segment z_η of Z, and works within $P \subseteq [z_\eta]$. It calls procedures $S_x^{e,\eta\alpha}(P, r, z_\eta)$ for certain x, α. In this case we let $z_{\eta\alpha} = y_\alpha = \mathsf{kuc}(P, r, z_\eta, \alpha)$.

For each x, once $J^A(x)\downarrow$, $R^{e,\eta}$ wishes to run $S_x^{e,\eta\alpha}$ for all α of a certain length m defined in (5) below, which increases with $h(x)$. Thus, as x increases, more and more bits beyond η are coded into Z. The trace set T_x will contain all the numbers enumerated by procedures $S_x^{e,\eta\alpha}$ where $|\alpha| = m$. We ensure that m is small enough so that $|T_x| \leq h(x)$. To summarize, a typical sequences of calls of procedures is

$$R^{e,\eta} \;\rightarrow\; S_x^{e,\eta\alpha} \;\rightarrow\; R^{e+1,\eta\alpha}.$$

Formal details. Some ML-random set $Y \not\geq_T \emptyset'$ is superhigh by pseudo jump inversion as in [7, 6.3.14]. Since $A \leq_T Y$ and A is c.e., A is a base for ML-randomness; see [7, 5.1.18]. Thus A is superlow. Hence there is an order function g and a computable enumeration of A such that $J^A(x)[s]$ becomes undefined for at most $g(x)$ times.

We build a sequence of Π^0_1 classes $(P^n)_{n\in\mathbb{N}}$ as in Lemma 1. If $n = \langle e, \gamma, x, i\rangle$, then since $K(n) \leq^+ 2\log\langle e, \gamma\rangle + 2\log x + 2\log i$, we have

$$P^{\langle e,\gamma,x,i\rangle} \neq \emptyset \;\Rightarrow\; \lambda P^{\langle e,\gamma,x,i\rangle} \geq 2^{-q} \tag{4}$$

where $q = 2\log\langle e, \gamma\rangle + 2\log x + 2\log i + c$ for some fixed $c \in \mathbb{N}$. By the Recursion Theorem we may assume that we know c in advance.

The construction starts off by calling $R^{0,\varnothing}(Q_0, 3, \varnothing)$.

Procedure $R^{e,\eta}(P, r, z)$, where $z \in 2^{<\omega}$, $P \subseteq \mathsf{MLR} \cap [z]$ is a Π^0_1 class and $r \in \mathbb{N}$. This procedure enumerates a c.e. trace $(T_x)_{x\in\mathbb{N}}$. (It assumes that $2^{-r} < \lambda P$.) For each string α of length at most the stage number s, see whether some procedure $S_x^{e,\eta\alpha}(P)$ requires attention, or is at (b) or (e), and no procedure $S_y^{e,\eta\beta}(P)$ for $\beta \prec \alpha$ satisfies the same condition. If so, choose x least for α and activate $S_x^{e,\eta\alpha}(P)$. (This suspends any runs $S_z^{e,\rho}$ for $\eta\alpha \preceq \rho$. Such a run may be resumed later.)

Procedure $S_x^{e,\eta\alpha}(P, r, z)$, where $|\alpha|$ is the greatest $m > 0$ such that, if $n = m(r + m + 1)$, we have

$$2^{|\eta\alpha|}2^{2n+r+2} \leq h(x). \tag{5}$$

There only is such a procedure if x is so large that m exists.

Let $y_{\alpha,s} = \mathsf{kuc}(P_s, \alpha, r, z)$. Let

$$\delta = 2^{-|y_{\alpha,s}|-m-r-1}.$$

(Comment: $S_x^{e,\eta\alpha}(P, r, z)$ cannot change $y_{\alpha,s}$. It only changes "by itself" as P_s gets smaller. This makes the procedure go back to the beginning. So in the following we can assume y_α is stable.)

Phase I.

(a) $S_x^{e,\eta\alpha}$ *requires attention* if $w = J^A(x)\downarrow$ with use u. Let

$$C = [y_\alpha] \cap \{Z\colon\ \Phi_e^Z\upharpoonright_u = A\upharpoonright_u\},$$

a Σ_1^0 class. Let $C_s = [y_{\alpha,s}] \cap \{Z\colon\ \Phi_e^Z\upharpoonright_u = A\upharpoonright_u [s]\}$ be its approximation at stage s, which is clopen.

(b) WHILE $\lambda(P_s \cap C_s) < \delta$ run in case $e < s$ the procedure

$$R^{e+1,\eta\alpha}(Q, q, y_{\alpha,s});$$

here Q is the Π_1^0 class $P \cap [y_{\alpha,s}] - C$, and

$$q = 2\log\langle e, \eta\alpha\rangle + 2\log x + 2\log i + c,$$

where i is the number of times $S_x^{e,\eta\alpha}$ has called $R^{e+1,\eta\alpha}$ (the constant c was defined after (4) at the beginning of the formal construction). Then $2^{-q} < \lambda Q$ unless $Q = \emptyset$. Meanwhile, if $y_{\alpha,s} \neq y_{\alpha,s-1}$ put w into T_x, cancel all sub-runs, GOTO (a), and RETURN. Otherwise, if $A_s\upharpoonright_u \neq A_{s-1}\upharpoonright_u$ cancel all sub-runs, GOTO (a) and RETURN.
(Comment: if the run $S_x^{e,\eta\alpha}$ stays at (b) and $Z \in Q$, then $A\upharpoonright_u = \Phi_e^Z\upharpoonright_u$ fails, so we have defeated Φ_e.)

(c) Put w into T_x, let $D = C_s$, GOTO (d), and RETURN. (Thus, the next time we call $S_x^{e,\eta\alpha}(P)$ it will be in Phase II.)

Phase II.

(d) $S_x^{e,\eta\alpha}$ requires attention again if $A\upharpoonright_u$ has changed.

(e) WHILE $P_s \cap D \neq \emptyset$ RUN in case $e < s$

$$R^{e+1}(P \cap D, q, y_{\alpha,s})$$

where $q \in \mathbb{N}$ is defined as in (b). Meanwhile, if $y_{\alpha,s} \neq y_{\alpha,s-1}$ cancel all sub-runs, GOTO (a), and RETURN.
(Comment: if the run $S_x^{e,\eta\alpha}$ stays at (e) and $Z \in Q$ then again $A\upharpoonright_u = \Phi_e^Z\upharpoonright_u$ fails, this time because $Z \in D$ and $\Phi_e^Z\upharpoonright_u$ is an old version of $A\upharpoonright_u$.)

(f) GOTO (a) and RETURN.

Verification. The function g was defined at the beginning of the formal proof. First we compute bounds on how often a particular run $S_x^{e,\eta\alpha}$ does certain things.
Claim 1. *Consider a run $S_x^{e,\eta\alpha}(P, r, z)$ called by $R^{e,\eta}(P, r, z)$. As in the construction, let $m = |\alpha|$ and $n = m(r + m + 1)$.*

(i) *While $y_{\alpha,s}$ does not change, the run passes (f) for at most 2^{m+r+1} times.*

(ii) The run enumerates at most 2^{2n+r+2} elements into T_x.

(iii) It calls a run $R^{e+1,\eta\alpha}$ at (b) or (e) for at most $2^{n+1}g(x)$ times.

To prove (i), as before let $\delta = 2^{-|y_\alpha|-m-r-1}$. Note that each time the run passes (f), the class $P \cap [y_\alpha]$ loses $\lambda D \geq \delta$ in measure. This can repeat itself at most 2^{m+r+1} times. (This argument allows for the case that the run of $S_x^{e,\eta\alpha}$ is suspended due to the run of some $S_z^{e,\eta\beta}$ for $\beta \prec \alpha$. If $S_z^{e,\eta\beta}$ finishes then $S_x^{e,\eta\alpha}$, with the same parameters, continues from the same point on where it was when it was suspended.)
(ii) There are at most 2^n values for y_α during a run of $S_x^{e,\eta\alpha}$ by the remarks after Definition 2. Therefore this run enumerates at most $2^n 2^{n+r+1} + 2^n$ elements into T_x where at most 2^n elements are enumerated when y_α changes.
(iii): for each value y_α there are at most $2g(x)$ calls, namely, at most two for each computation $J^A(x)$ (g is defined at the beginning of the formal proof). $\diamond$

Note that $|T_x| \leq h(x)$ by (ii) of Claim 1 and (5).

Strings $z_{\gamma,s}$, $\gamma \in 2^{<\omega}$ are used to code the given set G into Z'. Let $z_{\varnothing,s} = \varnothing$.

- If $z_{\eta,s}$ has been defined and $R^{e,\eta}(P, r, z_{\eta,s})$ is running at stage s, then for all β such that no procedure $S^{e,\eta\alpha}$ is running for any $\alpha \prec \beta$, let $z_{\eta\beta,s} = \mathsf{kuc}(P, r, z_{\eta,s}, \beta)$.
- If α is maximal under the prefix relation such that $z_{\eta\alpha,s}$ is now defined, it must be the case that $R^{e+1,\eta\alpha}(Q, q, z_{\eta\alpha})$ runs. So we may continue the recursive definition. Note that $|\alpha| > 0$ by the condition that $m > 0$ in (5).

Claim 2 *For each γ, $z_\gamma = \lim_s z_{\gamma,s}$ exists, with the number of changes computably bounded in γ.*

We say that a run of $S_x^{e,\rho}$ is a *k-run* if $|\rho| \leq k$. For each number parameter p we will let $\overline{p}(k, v)$ denote a computable upper bound for p computed from k, v. Such a function is always chosen nondecreasing in each argument.

To prove Claim 2, we think of k as fixed and define by simultaneous recursion on $v \leq k$ computable functions $\overline{r}(k, v), \overline{x}(k, v), \overline{b}(k, v)$, $\overline{c}(k, v)$ with the following properties:

(i) $\overline{r}(k, v)$ bounds r in any call $R^{e,\eta}(Q, r)$ where $|\eta| \leq k$ and $e \leq v$.
(ii) $\overline{x}(k, v)$ bounds the largest x such that some k-run $S_x^{e,\eta\alpha}$ is started where $e \leq v$.
(iii) For each x, $\overline{b}(k, v)$ bounds the number of times a k-run $S_x^{e,\eta\alpha}$ for $e \leq v$ requires attention.
(iv) For each x, $\overline{c}(k, v)$ bounds the number of times a run $R^{e+1,\eta\alpha}$ is started by some k-run $S_x^{e,\eta\alpha}$ for $e \leq v$.

Fix γ such that $|\gamma| = k$. In the following we may assume that $\eta\alpha \preceq \gamma$, because then the actual bounds can be obtained by multiplying with 2^k.

Suppose now $k \geq v \geq 0$ and we have defined the bounds in (i)–(iv) for $v-1$ in case $v > 0$. We define the bounds for v and verify (i)–(iv). We may assume $e = v$, because then the required bounds are obtained by adding the bounds for $k, v-1$ to the bounds now obtained for $e = v$.
(i). First suppose that $v = 0$. Then $\eta = \varnothing$, so let $\overline{r}(k, 0) = 3$. If $v > 0$, we define a sequence of Π^0_1 classes as in Lemma 1: if for the i-th time a run $S_x^{e-1,\rho}$ calls a run $R^{e,\rho}(Q, q)$ we let $P^{\langle e,\rho,x,i\rangle} = Q$. By the inductive hypothesis (iii)

and (iv) for $v-1$ we have a bound $\bar{i}(v,x)$ on the largest i such that a class $P^{\langle v,\eta\alpha,x,i\rangle}$ is defined (when $S_x^{v-1,\eta}$ in (b) or (e) starts a run $R^{v,\eta}$). Thus let $\overline{r}(k,v) = 2\log\langle v,\gamma\rangle + 2\log\overline{x}(k,v-1) + 2\log\bar{i}(v,\overline{x}(k,v-1)) + c$.
To prove (ii) and (iii), suppose $R^{e,\eta}(Q,r)$ calls $S_x^{e,\eta\alpha}$. Let $m = |\alpha|$ and $n = m(r+m+1)$. Then $n \le k(\overline{r}(k,v)+k+1)$.
(ii) We have $h(x) < 2^{k+2k(\overline{r}(k,v)+k+1)+3}$ because m is chosen maximal in (5). Since h is an order function, this gives the desired computable bound $\overline{x}(k,v)$ on x.

(iii). By Claim 1(i), for each value of y_α, the run can pass (f) for at most $2^{k+\overline{r}(k,v)+1}$ times. Further, it can require attention $2^n + g(\overline{x}(k,v))$ more times because y_α changes or because $J^A(x)$ changes. This allows us to define $\overline{b}(k,v)$.

(iv). By Claim 1(iv) a run $R^{v+1,\eta\alpha}$ is started for at most $\overline{b}(k,v)2^{k+1}g(\overline{x}(k,v))$ times.

This completes the recursive definition of the four functions. Now, to obtain Claim 2, fix γ. One reason that z_γ changes is that (A) some run $S_y^{e,\rho}$ for $\rho \preceq \gamma$, calls $R^{e+1,\rho}$ in (e). This run is a k-run for $k = |\gamma|$. By (ii) and (iii), the number of times this happens is computably bounded by $\overline{b}(k,k)\overline{x}(k,k)$. While it does not happen, z_γ can also change because (B) for some $\eta\alpha \preceq \gamma$ as in the construction, y_α changes because some P_s, which defines y_α, decreases. Since there is a computable bound $\bar{l}(k)$ on the length of z_γ by (i) of this claim and (3), while the first reason does not apply, this can happen for at most $2^{\bar{l}(k)}$ times. Thus in total z_γ changes for at most $\overline{b}(k,k)\overline{x}(k,k)2^{\bar{l}(k)}$ times. ◇

Now let $Z = \bigcup_{\gamma \prec G} z_\gamma$. By Claim 2 we have $G \le_{\mathrm{tt}} Z'$.
Claim 3 (Golden Run Lemma) *For some $\eta \prec G$ and some e, there is a run $R^{e,\eta}(P,r)$ (called a golden run) that is not cancelled such that, each time it calls a run $S_x^{e,\eta\alpha}$ where $\eta\alpha \prec G$, that run returns.*
Assume the claim fails. We verify the following for each e.

(i) There is a run $R^{e,\eta}$ that is not cancelled; further, $S_x^{e,\eta\alpha}(P)$ is running for some x, where $\eta\alpha \prec G$, and eventually does not return.
(ii) $A \ne \Phi_e^Z$.

(i) We use induction. For $e = 0$ clearly the single run of $R^{0,\varnothing}$ is not cancelled. Suppose now that a run of $R^{e,\eta}$ is not cancelled. Since we assume the claim fails, some run $S_x^{e,\eta\alpha}$, $\eta\alpha \prec G$, eventually does not return. From then on the computation $J^A(x)$ it is based on and y_α are stable. So the run calls $R^{e+1,\eta\alpha}$ and that run is not cancelled.
(ii) Suppose the run $S_x^{e,\eta\alpha}(P,r,z)$ that does not return has been called at stage s. Suppose further it now stays at (b) or (e), after having called $R^{e,\eta\alpha}(Q,q,y_\alpha)$. Since $y_{\eta\alpha}$ is stable by stage s, we have $Z \in Q$. Hence $A \ne \Phi_e^Z$ by the comments in (b) or (e). ◇

Let $(T_x)_{x\in\mathbb{N}}$ be the c.e. trace enumerated by this golden run.
Claim 4 *$(T_x)_{x\in\mathbb{N}}$ is a trace for J^A with bound h.*

As remarked after Claim 1, we have $|T_x| \le h(x)$. Suppose x is so large that m in (5) exists. Suppose further that the final value of $w = J^A(x)$ appears at stage t. Let $\eta\alpha \prec G$ such that $|\alpha| = m$.

As the run is golden and by Claim 1(i), eventually no procedure $S_y^{e,\eta\beta}(P)$ for $\beta \prec \alpha$ is at (b) or (e). Thus, from some stage $s > t$ on, the run $S_x^{e,\eta\alpha}$ is not suspended. If y_α has not settled by stage s then w goes into T_x. Else $\lambda(P \mid y_{\alpha,s}) > 2^{-r-|\alpha|}$. Since $S_x^{e,\eta\alpha}$ returns each time it is called, the run is at (a) at some stage after t. Also, $P_s \cap C_s$ must reach the size $\delta = 2^{-|y_\alpha|-|\alpha|-r-1}$ required for putting w into T_x.

As a consequence, we can separate highness properties within the ML-random sets. See [7, Def. 8.4.13] for the weak reducibility $\leq_{JT}$, and [10] for the highness property "$\emptyset'$ is c.e. traceable by Y". Note that JT-hardness implies both this highness property and superhighness.

Corollary 1. *There is a ML-random superhigh Δ^0_3 set Z such that $\emptyset'$ is not c.e. traceable by Z. In particular, Z is not JT-hard.*

Proof. By [7, Lemma 8.5.19] there is a benign cost function c such that each c.e. set A that obeys c is Turing below each ML-random set Y such that $\emptyset'$ is c.e. traceable by Y. By [7, Exercise 8.5.8] there is an order function h such that some c.e. set A obeys c but is not jump traceable with bound h. Then by the proof of Theorem 2 there is a ML-random superhigh set $Z \leq_T \emptyset''$ such that $A \not\leq_T Z$. Hence Z is not JT-hard.

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