The Unit Conjecture, Classes of Torsion Free Groups, and Connections to Logic

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1 Overview

This dissertation aims to discuss a number of interesting results about torsion free groups, explain Giles Gardam's counterexample to the Unit conjecture for group rings, and show how these results are connected to logic.

1.1 Introduction

The unit conjecture for group rings was originally stated in Higman's 1940 unpublished thesis [12] and was later popularized by Kaplansky. It claims that if K is a field and G is a torsion-free group, then the only units of $K[G]$ are the trivial units. Giles Gardam's counterexample to Kaplansky's unit conjecture offers a springboard to investigate certain group-theoretic properties related to the conjecture, and their relationship to logic.

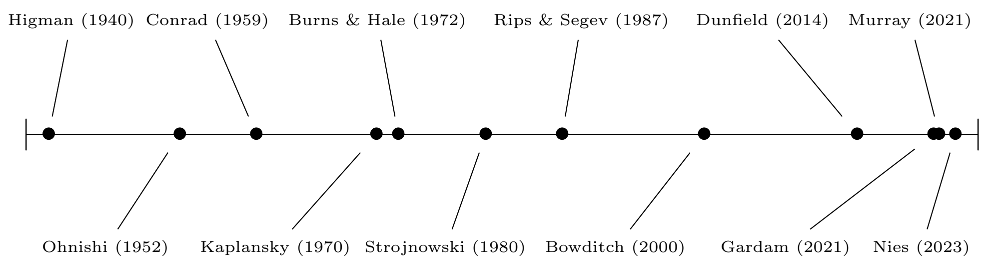
The second section introduces the relevant group properties and their relationships, as well as presenting proofs from Burns and Hale [4], Ohnishi [24], and Conrad [6], the latter's yielding a corollary that a group being one-sided orderable is expressible in first-order logic.

The third section introduces some preliminaries to understanding Gardam's paper, in which he presents the aforementioned counterexample [9]. We then explain some aspects of Gardam's paper, while including some requisite background knowledge to make reading the paper more accessible.

The fourth section explores how the group-theoretic properties from Section 2 relate to first-order logic, and explains Gardam's use of boolean logic to help obtain his result. We conclude by looking at a result by Alan G. Murray, which extends Gardam's counterexample by presenting a class of counterexamples falsifying the conjecture for every positive characteristic [20].

1.2 Timeline and History

Below is a timeline depicting some of the major papers drawn on in this project. The papers on this timeline fit into two distinct but related historical narratives. One, a search for results about orderable groups, the other, a hunt to resolve Kaplansky's conjectures.



Higman initially proposed the unit conjecture for group rings in his unpublished thesis (1940)[12]. The next point of interest for us were the results in the 1950s by Ohnishi (1952) [24] and Conrad (1959) [6], which established criterions for one-sided orderable groups. It is mentioned in the introduction of *Groups, Orders and Dynamics* (2016) that the study of bi-orderable groups begins with the “seminal works of Dedekind, Hölder, and Hilbert”[8]. In 1970, kaplansky popularized the conjectures about group rings which now bear his name [16]. In 1972, Burns and Hale published a paper detailing some results about one-sided orderable groups and groups with the unique product property, which used Conrad’s criterion [4].

Later, Strojnowski (1980) proved that groups with the unique product property also have the two-unique products property [30], and 7 years later Rips and Segev (1987) presented a torsion-free group without unique products [26]. Much later, in 2000, Bowditch defined the term diffuse, which acts as a geometric analogue of the unique products property [3]. In the next decade, Dunfield (2014) gave an example of a diffuse group which is not one-sided orderable [17]. Then in 2021, Gardam published his counterexample [9]. Later that year Murray (2021) had generalized it to characteristic p for all primes, p [20]. Now in 2023, Nies (in part with Gardam’s observations) has shown that many of the properties integral to these results are expressible in first-order logic [22].

The bulk of the material in this project is drawn from Gardam’s 2021 paper and Nies’ Logic Blog. There are many other results and counterexamples related to the above narrative which I have mentioned throughout the paper. Those included above are either milestone papers in the narrative of group rings, ordered groups, and Kaplansky’s conjectures or are papers I have drawn on heavily, or both.

2 Groups and Group Rings

2.1 Group-Theoretic Properties

In this section, we will introduce and discuss the relationship between group-theoretic properties which have been relevant in the search for a resolution to the unit conjecture.

Definition 2.1.1. (Kaplansky [16], Passman [25]) A group G has the *unique product property* if, for nonempty sets $A, B \subseteq G$, there is $(a, b) \in A \times B$ such that for all $a_1, b_1 \in A \times B$ with $a_1 \neq a$ and $b_1 \neq b$ that $ab \neq a_1 b_1$.

Definition 2.1.2. A group G has the *two unique products property* if it has the unique product property and if A, B are not singletons then there are exactly two pairs $(a_1, b_1), (a_2, b_2) \in A \times B$ so that for some $ab \in AB$, we have $ab = a_1 b_1$ and $ab = a_2 b_2$.

In 1980, Strojnowski published a proof that groups with the unique product property have the two unique products property [30].

Definition 2.1.3. (Higman [12], Burns, Hale [4]) Let, X denote a class of groups closed under taking isomorphic images. A group G is locally X-indicable if every nontrivial finite subgroup of G can be mapped homomorphically onto a nontrivial group in X.

f.g.

Theorem 2.1.4. *Let Ω denote the class of groups with the unique product property. If a group is locally Ω-indicable, then it is in Ω.*

Proof. Rephrasing of Theorem 1 in [4]. Suppose that G is locally Ω-indicable, but is not in Ω. Let A and B be nonempty finite subsets of G without the unique product property, so for each pair (a, b), there exists (a_1, b_1) with $ab = a_1 b_1$ and $a \neq a_1$, $b \neq b_1$. Suppose also that $|A| + |B|$ is minimal. Note that we can assume $1_G \in A, B$, because if $A = \{a_1, \dots, a_k\}$ and $B = \{b_1, \dots, b_d\}$ we can instead consider (without loss of generality) $a_1^{-1}A = \{1, \dots, a_1^{-1}a_k\}$ and $b_1^{-1}B = \{1, \dots, b_1^{-1}b_d\}$. Clearly the size of each set is preserved, it is not hard to check that they still don't have unique products.

Let $H = \langle A, B \rangle$. Because G is locally Ω-indicable, there exists a homomorphism $\phi : H \to W$, where $W \in \Omega$. Since Ω is closed under taking subgroups, it follows by the 1st isomorphism theorem that $H/K \cong M \leq W$ where $K = ker\phi$ and $M \in \Omega$. So then there is $K \triangleleft H$ such that $H/K \in \Omega$. Let $\phi : H \to H/K$ be the canonical homomorphism. Because $\phi(A), \phi(B)$ are nonempty subsets of H/K, they contain Ka, Kb which clearly have unique products.

Let $A_1 = A \cap Ka$ and $B_1 = B \cap Kb$. Then A_1 and B_1 do not have unique products, as, if they did then, since $a_1 \notin Ka$ and $b_1 \notin Kb$ then $a_1b_1 \notin KaKb$, and so $ab \neq a_1b_1$. Additionally, if $A_1 = A$ and $B_1 = B$, then $A \subseteq Ka$ and $B \subseteq Kb$. Then, as $1 \in A, B$ we get $Ka = Kb = K$ which implies H/K is trivial. Therefore $|A_1|+|B_1| < |A|+|B|$, so there is a smaller counterexample which contradicts minimality.

□

Definition 2.1.5. A group G with elements totally ordered under $\leq$ is *one-sided orderable* if it is invariant under right or left multiplication (also called translation invariant): $a \leq b \to ac \leq bc$. A group which is both left and right invariant under multiplication with respect to the same ordering, is called bi-orderable

A group being right orderable is equivalent to it being left orderable (hence the term one-sided orderable), suppose a totally ordered group $(G, \leq_1)$ is right orderable. We define a new ordering $a \leq_2 b$ if and only if $b^{-1} \leq_1 a^{-1}$. For our purposes, in proofs and computations, we will often take a one-sided orderable group to be right orderable.

Additionally, if a group is one-sided orderable, it is torsion-free, because if a is a non identity element with $1 \leq a$, then $1 \leq a \leq a^2 \leq \ldots$ and if $1 \geq a$, then $1 \geq a \geq a^2 \geq \ldots$

The converse is not true. A counterexample is the group given by presentation: $\langle a, b \mid (a^2)^b = a^{-2}, (b^2)^a = b^{-2} \rangle$ [8], which we will meet again later under the name P.

Example 2.1.6. Dehornoy has shown that braid groups are one-sided orderable by describing an explicit ordering, see [7].

Braid groups can be described algebraically. We denote the Artin braid Group on $n-1$ generators, $\sigma_1, \ldots, \sigma_{n-1}$, as B_n. The relators are of the form

$$\sigma_i\sigma_j = \sigma_j\sigma_i \qquad \text{and} \qquad \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$$

for $1 \leq i, j \leq n-1$ and $1 \leq i \leq n-2$ respectively. Note that $B_2 \cong (\mathbb{Z}, +)$.

Example 2.1.7. Torsion-free nilpotent groups are bi-ordered. A short proof can be found on page 15 in Section 1.2.1 of [8].

The below definition of a *diffuse* group was given by Bowditch in [3]. His stated goal in defining diffuse groups was to "understand the unique product property of groups from a more geometric point of view..."

Definition 2.1.8. (Bowditch [3]) A group G is *diffuse*, if for all nonempty finite sets $C \subseteq G$, there exists $c \in C$, called an extremal element, such that for $g \in G \setminus \{1\}$ one has $gc \notin C$ or $g^{-1}c \notin C$.

Example 2.1.9. Any one-sided orderable group is diffuse. let C be a finite subset of some one-sided orderable group G, and let $c = \max(C)$. Then consider, $k \neq 1$. Without loss of generality, assume $kc \in C$, then because c is maximal, and $k \neq 1$ it follows $kc < c$. But because G is one-sided orderable, $c < k^{-1}c$. So then $k^{-1}c \notin C$.

In 2014, Dunfield presented a group which is diffuse but not one-sided orderable. Dunfield's counterexample is a topological construction, specifically the fundamental group of a closed orientable hyperbolic 3-manifold, M. Its fundamental group, denoted $\pi_1(M)$, has presentation $\langle a, b \mid a^2b^{-1}a^2b^2a^{-1}b^2 = 1, ab^2a^{-1}ba^{-2}ba^{-1}b^2 = 1\rangle$ [17].

We have seen that G being one-sided orderable implies that G is diffuse. It can be shown that diffuse groups have the unique product property, a proof can be found on page 3 of [22]. However it is also possible to prove directly that one-sided orderable groups have unique products, was was first shown by Botto Mura and Rhemtulla in 1975 [2].

Lemma 2.1.10. *One-sided orderable groups have unique products.*

Proof. This proof closely follows the one in Gardam's slides [11]. Consider two finite subsets A, B of a one-sided orderable group G, such that their members are ordered as below:

$$a_1 \leq a_2 \leq \ldots \leq a_m$$

$$b_1 \leq b_2 \leq \ldots \leq b_n$$

We have, by right-invariance, that $a_ib_1 \leq a_ib_j$ for all i and for $j \neq 1$, since $b_1 \leq b_j$. This means a_ib_1 is minimal for each i. Consider these minimal elements $a_1b_1, \ldots, a_mb_1$. These must all be distinct, otherwise we would find at least one pair a_l,a_k with $a_l = a_k$ via cancellation. Since there must also exist a minimum product (A and B are finite) then there exists a unique product among these elements.

□

2.2 Linearly Ordered Groups

This subsection will present some results concerning linearly ordered groups proved by Masao Ohnishi and Paul Conrad. In his paper *Linear-Order on a Group*, Ohnishi uses the term *linear order*, and in lieu of a definition, provides a citation to the paper *On linearly ordered groups* by Kenkichi Iwasawa (1948), in which a linear order on a group is defined as a total order which is bi-invariant [14]. It seems "linearly ordered group" has since taken on a more general meaning, and can now mean bi-orderable or one-sided orderable. In Section 4, it is shown that Conrad's result allows for a nice expression for "one-sided orderable" in first-order logic. However, we should first introduce semigroups, which appear in the major proofs of this section.

Definition 2.2.1. A *semigroup* is set S equipped with, and closed under, an associative binary operation $S \times S \to S$. We will denote, by $(x_1, \dots, x_n)$, the semigroup obtained from all combinations of $x_1, \dots, x_n$, under such a binary operation. We will call $(x_1, \dots, x_n)$ the semigroup *generated* by $x_1, \dots, x_n$. The shorthand $(x_i \mid 1 \leq i \leq n)$ may also be used, to denote the same object.

Example 2.2.2. Up to isomorphism, there are five semigroups on two elements. Three of these are:

(I) The integers under addition mod 2, $(\mathbb{Z}_2, +)$. We know it is a group, and every group is also a semigroup.

(I) The right-zero semigroup, $R0_2$, of order 2. In a right-zero semigroup, S, we have that for all $a, b \in S$, $ab = b$. $R0_2$ has no identity element, and so is not a group.

(III) Logical AND over the truth values 0 (false) and 1 (true) denoted $(\{0, 1\}, \wedge)$ which has the following Cayley table:

$\wedge$	0	1
0	0	0
1	0	1

We see that $(\{0, 1\}, \wedge)$ fails to be a group. $\wedge$ is clearly an internal associative binary operation. Additionally, there exists an identity element 1, making this semigroup also a monoid. However, 0 does not have an inverse, so it fails to satisfy all group axioms.

Remark. Choosing logical OR as the group operation instead would also produce a semigroup. However this cannot be extended to all logical operators. For example, logical implication is not associative.

The two remaining semigroups are the left-zero semigroup of order 2 and the null semigroup. These examples are all finite semigroups, however, semigroups can be infinite and there is an infinite example which will be of particular relevance in the next section. The following is taken from Rotman's *An Introduction to the Theory of Groups*.

Definition 2.2.3. (Rotman, page 349) If Σ is a semigroup and $X \subseteq \Sigma$, then Σ is a *free semigroup* with basis X if for every semigroup S and every function $f : X \to S$, there exists a unique homomorphism $\varphi : \Sigma \to S$ extending f [27].

Example 2.2.4. The *words* on a set X are strings consisting of the elements of X formed by concatenation. For a formal treatment and construction of free groups and words, see Chapter 11 of the above book by Rotman. A word w over X is positive if either $w = 1$ or $w = x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n}$ where each ϵ_i for $1 \leq i \leq n$ is positive. In this case, n is the *length* of the word. The set of all positive words on x is a free semigroup with basis X [27].

We now establish the definitions and lemmas required for Ohnishi and Conrad's proofs.

Definition 2.2.5. An *invariant subsemigroup* is a subset $S \subseteq G$ which is closed under the operation of G, and is closed under conjugation by elements from G. That is, if $s \in S$ and $g \in G$ then $gsg^{-1} \in S$.

Definition 2.2.6. (Ohnishi) An *ordering set* is a subset H of G fulfilling the following properties.

(1) H is an invariant subsemigroup of G and $1 \in H$

(2) H contains either x or x^{-1} for all $x \in G$.

Lemma 2.2.7. *An ordering set, H, of a group G determines a bi-order on G. If we remove the condition that H is invariant, then it instead defines a one-sided order.*

Proof. Define an ordering: $a \leq b$ if and only if $ba^{-1} \in H$. Then we see that $bcc^{-1}a^{-1}$ if and only if $ac \leq bc$, hence G is one-sided orderable. If H is invariant, we have that $cba^{-1}c^{-1} \in H$ but then, by the ordering defined above, we have $ca \leq cb$, so G is bi-orderable. □

Definition 2.2.8. A family of sets F, is of finite character if and only if, for all $A \in F$, every finite subset of A is in F.

The third form of Zorn's lemma given on page 7 of [31] states that given a set and a property of finite character, there exists a maximal subset having that property.

Theorem 2.2.9. *(Ohnishi) The following are equivalent:*

(I) *G is bi-orderable.*

(II) *for every finite subset $\{x_1, \ldots, x_n\}$ of G, we have*

$$\bigcap_{\epsilon \in \{1,-1\}^n} (1, x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n}) = 1$$

(III) *For any element $a \in G$, there is an ordering set S_a containing a and having the property that if some non-trivial product $xy \in S_a$, then either $x \in S_a$ or $y \in S_a$* [24].

Proof. (I) $\implies$ (II): Suppose G is bi-orderable, then for every element $x \in G$ we get $1 \leq x$ or $1 \leq x^{-1}$, because $x^{-1} \leq 1$ means $x^{-1}x \leq x$ so $1 \leq x$. Hence, for the choice of $\epsilon_i \in \{-1, 1\}$ which results in $1 \leq x_i^{\epsilon_i}$, we have that if $1 \leq x_i$ and $1 \leq x_j$, then, by translation invariance, $1 \leq x_i \leq x_i x_j$ and so by induction on the number of elements $1 \leq \min(1, x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n})$. By a symmetric argument, $\max(1, x_1^{-\epsilon_1}, \ldots, x_n^{-\epsilon_n}) \leq 1$. Therefore $(1, x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n}) \cap (1, x_1^{-\epsilon_1}, \ldots, x_n^{-\epsilon_n}) = 1$.

(II) $\implies$ (III): Let $X_{i \in I}$, for some index set I, be a family of subsets of G such that the following are satisfied for all $i \in I$:

(1) X_i is an invariant subsemigroup containing 1.

(2) For any finite subset $\{x_1, \ldots, x_n\}$ of G and some $a \in G$ we have:

$$a \notin \bigcap_{\epsilon \in \{1,-1\}^n} (X_i, x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n})$$

Because we are assuming (II), we have that $X_{i \in I}$ is nonempty. Furthermore, the collection of subsets with the above properties has finite character. This is because if $(S, \cdot)$ is a semigroup, and $H \subseteq S$, then $(H, \cdot)$ is also, and invariance is likewise inherited. Additionally, if the semigroup generated by S does not contain a and $H \subseteq S$, then neither does the semigroup generated by H. So then we can use Zorn's lemma (as stated in [31]) to deduce that there is some maximal subset, X_m, of G with the above properties.

We know X_m must contain either x or x^{-1} for every $x \in G$. Suppose not, then $X_m \subset (X_m, x)$ and $X_m \subset (X_m, x^{-1})$ and so, as X_m is maximal, (X_m, x) and (X_m, x^{-1}) must fail property (2). Therefore there are $\{y_1, \ldots, y_k\}$ and $\{z_1, \ldots, z_r\}$ so that

$$\bigcap_{\alpha \in \{-1,1\}^k} (X_m, x, y_1^{\alpha_1}, \ldots, y_k^{\alpha_k})$$

$$\bigcap_{\sigma \in \{-1,1\}^r} (X_m, x^{-1}, z_1^{\sigma_1}, \ldots, z_r^{\sigma_r})$$

both contain a. Then of course

$$\bigcap_{\lambda \in \{-1,1\}^{k+r+1}} (X_m, x^{\lambda_0}, y_1^{\lambda_1}, \ldots, y_k^{\lambda_k}, z_1^{\lambda_{k+1}}, \ldots, z_r^{\lambda_{k+r}})$$

also contains a and so X_m fails property (2). So then X_m must contain x or x^{-1} for all $x \in G$.

Let $\overline{X_m} = G \setminus X_m$. Clearly $a \in \overline{X_m}$ and if $xy \in \overline{X_m}$ then one of x or y must be in $\overline{X_m}$ as otherwise $x, y \in X_m$ which is a semigroup, hence xy would be in X_m which is a contradiction. So we see $\overline{X_m}$ is our desired set satisfying (III), meaning $S_a = \overline{X_m}$.

(III) $\implies$ (I): By assumption, and with application of Zorn's lemma, there is a maximal S_M. We assume there is some $b \in G$ so that $b \notin S_M$ and $b^{-1} \notin S_M$. let S_b also satisfy (III), and let $b \in S_b$.

Now consider $Y = (S_M \cup S_b) \setminus S_M^{-1}$. Where $S_M^{-1} = \{x^{-1} \mid x \in S_M\}$. We see $b \in Y$, because $b \notin S_M$ and $b^{-1} \notin S_M$ (hence $b \notin S_M^{-1}$) and $b \in S_b$. Suppose $xy \in Y$. If xy is in S_M then x or y is in S_M, and the same for S_b as both satisfy (III). Therefore Y satisfies (III) and thus contradicts the maximality of S_M. So then S_M must contain x or x^{-1} for every element in G.

Consequently, S_M satisfies one requirement to be an ordering set. We now show it is a subsemigroup, by arguing it is closed. This is because if $a \in S_M$ and $b \in S_M$ then $ab \in S_M$ as otherwise, necessarily $b^{-1}a^{-1}$ would be in S_M and thus either a^{-1} or b^{-1} would be in S_M, a contradiction. We know from Lemma 2.2.7 that an ordering set determines a linear order. Therefore G is bi-orderable.

$\square$

In Conrad's paper, he says that the equivalences of properties (1), (2) and (3) in his Theorem 2.2 are proved exactly as in Ohnishi's paper but for omitting every occurrence of the word "invariant", this follows from Lemma 2.2.7.

Theorem 2.2.10. Conrad's Criterion. *A group is one-sided orderable if and only if for every identity-free finite subset $\{x_1, .., x_n\}$ of G there exists $\epsilon_i \in \{1, -1\}$ such that $1 \notin (x_1^{\epsilon_1}, ..., x_n^{\epsilon_n})$.*[6]

For ease, we will call the property right of the if and only if, (IV), so the above statement says (I) $\iff$ (IV).

Proof. We show (I) from Theorem 2.2.9 implies (IV) and that (IV) implies (II).

(I) $\implies$ (IV): Suppose G is one-sided orderable. Then, for all $\epsilon_i \in \{-1, 1\}$ with $1 \leq i \leq n$ we choose ϵ_i so that $x_i^{\epsilon_i} > 1$, then all elements in $(x_1^{\epsilon_1} \ldots, x_n^{\epsilon_n})$ are greater than 1, and so $1 \notin (x_i^{\epsilon_i})$.

(IV) $\implies$ (II): Suppose G does not satisfy (II), but does satisfy (IV). Then:

$$\bigcap_{\epsilon \in \{1,-1\}^n} (x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n}) \neq 1$$

So there must be some non-trivial element, a, in the above intersection. Then $a \in (x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n})$ and $a \in (x_1^{-\epsilon_1}, \ldots, x_n^{-\epsilon_n})$. Therefore, we have $a = \prod_{i \in I} x_i^{\epsilon_i}$ and $a = \prod_{j \in J} x_j^{-\epsilon_j}$, for $\epsilon_i \in \{\epsilon_1, \ldots, \epsilon_n\}$ and $\epsilon_j \in \{-\epsilon_1, \ldots, -\epsilon_n\}$ and for some (not necessarily equal) $I, J \subseteq \{1, \ldots, n\}$.

Hence $a^{-1} = \prod_{j \in J} x_j^{\epsilon_j}$. But then a and a^{-1} would be in $(x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n})$ and so the set $\{a, x_1, \ldots, x_n\}$ fails (IV). So by contradiction, we get the desired implication. □

Corollary 2.2.11. *If every finitely generated subgroup of G is one-sided orderable, then G is one-sided orderable.*

Proof. We prove the contrapositive. Suppose G is not one-sided orderable. Then, by Conrad's criterion, there is some finite subset $\{x_1, \ldots, x_n\}$ such that, for all $\epsilon_i \in \{1, -1\}$ we have $1 \in (x_1^{\epsilon_1}, \ldots, x_n^{\epsilon_n})$.

Let the subgroup $X = \bigcap_{i \in I} H_i$ where $H_{i \in I}$ is the family of all subgroups of G containing X. This family of subgroups is nonempty because G contains X. It is clear that X also fails Conrad's criterion, and hence is not one-sided orderable. □

This next result (Theorem 2 in [4]), by R.G. Burns and V.W.D Hale, uses Conrad's criterion to obtain a result about one-sided orderable groups and the property of being locally X-indicable (Definition 2.1.3).

Theorem 2.2.12. *Denote the class of groups which are one-sided orderable as Δ. If a group G is locally Δ-indicable, then $G \in \Delta$.*

Proof. (Adapted from Theorem 2 in [4]) Suppose G is a locally Δ-indicable group but $G \notin \Delta$. Then, by Conrad's criterion, there is a smallest minimal counterexample $M = \{g_1, \ldots, g_k\}$ so that $1 \notin M$, but $1 \in (g_1^{\epsilon_1}, \ldots, g_k^{\epsilon_k})$ for all $\epsilon_i \in \{1, -1\}$. Note that $1 < k < n$. Let $H = \langle M \rangle$. As in the previous proof, we can take $K \trianglelefteq H$ such that H/K is nontrivial and one-sided orderable.

We cannot have $Kg_i = K$ for all i, as this makes H/K trivial. However, we also cannot have $Kg_i \neq K$ for all i. We have, by assumption that $1 \in (g_i^{\epsilon_i})$, which yields $K \in (Kg_i^{\epsilon_i})$. But then, by Conrad's criterion, H/K would not be one-sided orderable. This would mean $K \in \{Kg_i\}$. So we instead suppose that the elements of M not in K are (relabelling if necessary) $g_1, \ldots, g_r$.

We know $H/K \in \Delta$ means there are $\mu_i \in \{1, -1\}$ so that $K \notin (Kg_i^{\mu_i} \mid 1 \leq i \leq r)$. Fix these μ_i. Because $r > 0$ but M is minimal, the smaller sub-semigroup $(g_i^{\mu_i} \mid r+1 \leq j \leq k)$ must be one-sided orderable, hence, must not contain 1 for some choices of μ_j. Now because M is not one-sided orderable, 1 can be written $1 = g_b^{\mu_b} \cdot \ldots \cdot g_c^{\mu_c}$ for all choices of μ_l where $1 \leq b \leq l \leq c \leq k$.

Note that because $(g_i^{\mu_i} \mid r+1 \leq j \leq k)$ does not contain 1, for our choices of μ_j, we can infer that some element in the above expression for 1, has index $\leq r$. Multiplying 1 by K we get $K = Kg_b^{\mu_b} \cdot \ldots \cdot Kg_c^{\mu_c}$ (again, for all μ_l with $1 \leq b \leq l \leq c \leq k$) but because the elements $g_{r+1}, \ldots, g_k$ are definitionally all elements in M and in K, these disappear in the product (taking with them all of the μ_l for $r+1 \leq l \leq k$, and leaving the μ_l for $1 \leq l \leq r$), and thus, we can choose the remaining μ_l to be the μ_i we fixed earlier, and so deduce that K can be expressed by elements from $(K_{g_i}^{\mu_i} \mid 1 \leq i \leq r)$, which is a contradiction. □

Because we know the integers are one-sided orderable, a corollary to the above theorem is that a locally $\mathbb{Z}$-indicable (also called locally indicable) group is one-sided orderable.

Example 2.2.13. Let r be an element of the free group $F(X)$ on the generating set X. A *one-relator group* is a group with a presentation of the form $\langle X \mid r \rangle$. A nontrivial element, g, of a group, G, is called a *proper power* if there exists $h \in G$ and some integer $n > 1$ such that $h^n = g$. It was proved by Howie in 1980 that a one-relator group where r is not a proper power is locally $\mathbb{Z}$-indicable [13].

2.3 Group Rings

Kaplansky's conjectures concern group rings, which we now define. Some of the group-theoretic properties introduced in the last section turn out to be relevant to group rings and Kaplansky's conjectures.

Definition 2.3.1. Let G be a group and K a field. A *group ring* $K[G]$ is the ring of finite formal sums with coefficients from K. The underlying set is therefore $\{\sum_{g \in G} k_g g \mid k_g \in K, g \in G\}$ which is closed under the following operations.

$$\text{Addition} : (\sum_{g \in G} k_g g) + (\sum_{h \in G} k_h h) = \sum_{g,h \in G} (k_g g + k_h h)$$

$$\text{Multiplication} : (\sum_{g \in G} k_g g)(\sum_{h \in G} k_h h) = \sum_{g,h \in G} (k_g k_h gh)$$

$$\text{Scalar Multiplication} : m(\sum_{g \in G} k_g g) = (\sum_{g \in G} m k_g g)$$

Example 2.3.2. Consider the field $\mathbb{F}_2$ and the group $D_4 = \langle r, s \mid r^4 = s^2 = 1, srs = r^{-1} \rangle$. Then an example of a computation in the group ring $\mathbb{F}_2[D_4]$ is as follows:

$$(r^2 + s + 1)(r^2 + s) = r^4 + r^2 s + r^2 s + s^2 + r^2 + s = r^2 + s$$

Example 2.3.3. Group rings are similar to *polynomial rings*, $K[X]$, in X over a field K, which is the set of expressions:

$$p = k_0 + k_1 X + k_2 X^2 + \ldots + k_n X^n$$

Where the usual rules for adding, multiplying polynomials apply. X is a constant symbol, not in R. Polynomial rings are not group rings. Note that exponents on X are not allowed to be negative, and so the set $\{1, X, X^2, X^3, \ldots\}$ with the binary operation $X^n X^m = X^{n+m}$, does not form a group.

Example 2.3.4. The *Laurent polynomials* over some field K form a group ring. Usually denoted $K[X, X^{-1}]$. Negative exponents are allowed, and function as usual. So we see that the group ring $K[\mathbb{Z}]$ is exactly the Laurent polynomials over K.

Definition 2.3.5. In the following, R denotes a ring with unity.

$x \in R$ is a *unit* if $\exists y \in R$ such that $xy = yx = 1$.

$x \in R$ is a *zero divisor* if $\exists y \in R$ with $y \neq 0$, such that $xy = 0$.

$x \in R$ is *idempotent* under a binary operation, $\cdot$, if $x \cdot x = x$.

If k is a unit in the field K, and $g \in G$, then kg is called a *trivial unit* in $K[G]$.

Lemma 2.3.6. *Let K be a field, and G a group. If G has a torsion element, then the group ring $K[G]$ has nontrivial zero divisors.*

Proof. Let $x \in G$ and suppose there exists $n > 1$ such that $x^n = 1$. Then $(x-1)(1+x+\ldots+x^{n-1}) = (x+\ldots+x^{n-1}+1) - (1+x+\ldots+x^{n-1}) = 0$ and so $(x-1)$ is a nontrivial zero divisor. □

Definition 2.3.7. A ring, R, is called *prime* if $a, b \in R \setminus \{0\}$ implies there exists an $r \in R$ such that $arb \neq 0$.

Lemma 2.3.8. *G has no non-trivial, finite, normal subgroup if and only if $K[G]$ is prime* (Theorem 1 in [25]).

The following theorem outlines the implication structure among Kaplansky's conjectures.

Theorem 2.3.9. *Let K be a field and G a torsion-free group. Then, for the group ring $K[G]$ we have the following implications: $K[G]$ has only trivial units $\implies$ $K[G]$ has no zero divisors $\implies$ $K[G]$ has no idempotents other than 0 and 1.*

Proof. For the first implication, because G is torsion-free and all finite subgroups of a torsion-free group are trivial, we can apply Lemma 2.3.8, and find $K[G]$ is prime. Suppose, for the contrapositive, that $ab = 0$ in $K[G]$ where a or b is nonzero. Because $K[G]$ is prime, we can find c, such that $bca \neq 0$. Define $\alpha := bca$, then $\alpha^2 = 0$ and $(1-\alpha)(1+\alpha) = (1+\alpha)(1-\alpha) = 1$, meaning $1-\alpha$ is a unit. Suppose it was trivial, then $1-\alpha = kg$ so $\alpha = kg - 1$, which is in the subring $K[\langle g \rangle]$, G being torsion-free gives us that $\langle g \rangle \cong \mathbb{Z}$. So $kg - 1 \in K[\mathbb{Z}]$. Because this is a domain (which we get from $\mathbb{Z}$ being a domain), we have a contradiction.

For the second implication. We again prove the contrapositive. Simply note that any nontrivial idempotent, x, is a zero divisor, as $x(x-1) = 0$. □

3 Gardam's Counterexample

This section will explain some aspects of Giles Gardam's 2021 paper *A Counterexample to the Unit Conjecture for Group Rings.* We now restate the unit conjecture introduced in the last section.

If K is a field and G is a torsion-free group, then the only units of $K[G]$ are the trivial units.

3.1 The Group P and Requisite Definitions

Gardam's counterexample is located in the group given by this presentation.

$$P = \langle a, b \mid (a^2)^b = a^{-2}, (b^2)^a = b^{-2} \rangle$$

Where the exponents denote right conjugation, for example $b^a = a^{-1}ba$. Gardam informs us that P is also known as Hantzsche–Wendt group, Promislow group, and the Fibonacci group $F(2, 6)$ [9].

Definition 3.1.1. A sequence of homomorphisms $G_0 \xrightarrow{\varphi_1} G_1 \xrightarrow{\varphi_2} \dots \xrightarrow{\varphi_n} G_n$ mapping between groups is called *exact* if $im\varphi_i = ker\varphi_{i+1}$ at G_i, for each G_i in the sequence. If we consider the following exact sequence

$$1 \to N \xrightarrow{\iota} G \xrightarrow{\varphi} Q \to 1$$

we will find that ι is injective and φ is surjective. An exact sequence of this form is called a *short exact sequence*. In this case, we will call G an *extension* of Q by N, (however another convention is to say G is an extension of N by Q). Further, the above short exact sequence would be called *split* if $G \cong N \rtimes Q$ (where $\rtimes$ denotes a semidirect product - see page 167 in [27]) otherwise it is called *non-split*.

Example 3.1.2. The Group P introduced above is a non-split extension

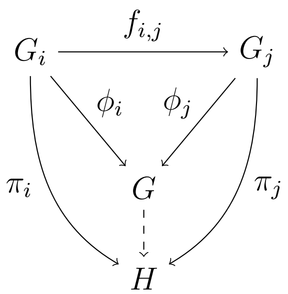
$$1 \to \mathbb{Z}^3 \xrightarrow{\iota} P \xrightarrow{\varphi} \mathbb{Z}_2 \times \mathbb{Z}_2 \to 1$$

Because ι is injective we can say $\mathbb{Z}^3 \cong \iota(\mathbb{Z}^3) \trianglelefteq P$. We also know that $ker\varphi = \iota(\mathbb{Z}^3)$ and later we will give the explicit map φ with kernel $\langle a^2, b^2, abab \rangle \cong \mathbb{Z}^3$. Note also that by the first isomorphism theorem $P/\mathbb{Z}^3 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, and this quotient group will later be important, for separating the elements of P into cosets.

Definition 3.1.3. In the category of groups (see section 1.1 of *Basic Category Theory* by Leinster for an introduction to categories [18]), a *direct system* is a directed set $(I, \leq)$ (where $\leq$ is a preorder such that every pair of elements has an upper bound) a family of groups $\{G_i \mid i \in I\}$ and a collection of homomorphisms $\{f_{ij} : G_i \to G_j \mid i \leq j\}$, satisfying the following properties:

- G_{ii} is the identity $id : G_i \to G_i$
- $f_{ik} = f_{jk} \cdot f_{ij}$ for all $i \leq j \leq k$.

Definition 3.1.4. The *direct limit* of a direct system can be denoted simply $\lim\limits_{\rightarrow} G_i$ and is defined as $\bigsqcup_i G_i/R$ where R is some equivalence relation. We can also show that this is equivalent to the existence of the unique dotted arrow which makes this diagram commute for all $i, j \in I$, where H is any group.



Definition 3.1.5. Suppose we are given a group A, a family of groups $(G)_{i \in I}$ and for each $i \in I$ an injective homomorphism $A \to G_i$. We identify A with its image in each of the G_i. We denote $*_A G_i$ the direct limit of the family (A, G_i) with respect to these homomorphisms, and call it the *product* of G_i with A *amalgamated* (see section 1.2 of Serre's *Trees* [28]).

When A is trivial, we say $*G_i$ is the *free product* of the G_i. Notation: sometimes the images which are being identified, are put as the subscript of $*$, e.g. $G_1 *_{f(\alpha)=g(\beta)} G_2$.

Example 3.1.6. In the case of the aforementioned group P, $A = \mathbb{Z}^2 = \langle (1,0), (0,1) \rangle$, $G_1 = \langle x, b \mid x^b = x^{-1} \rangle$, $G_2 = \langle y, a \mid y^a = y^{-1} \rangle$ and the functions $f_1 : A \to G_1$ and $f_2 : A \to G_2$ are given by:

$$f_1((1,0)) = x \qquad f_2((1,0)) = a^2$$
$$f_1((0,1)) = b^2 \qquad f_2((0,1)) = y$$

We set the additional relations $f_1(1,0)f_2(1,0)^{-1} = 1$ and $f_2(0,1)f_2(0,1)^{-1} = 1$. Hence setting the isomorphic subgroups $\langle x, b^2 \rangle$ and $\langle a^2, y \rangle$ equal to each other. Giving the following presentation which Gardam also provides:

$$\langle x, b, y, a \mid x^b = x^{-1}, y^a = y^{-1}, x = a^2, y = b^2 \rangle$$

Observe that this is just $\langle X_1 \cup X_2 \mid R_1 \cup R_2 \cup \{f_1(1,0) = f_2(1,0), f_2(0,1) = f_2(0,1)\}\rangle$. Where X_1 and R_1 are, respectively, the generators and relations of G_1 and likewise for X_2, R_2 and G_2. This shows P is an amalgamated free product of two Klein bottle groups.

Lemma 3.1.7. *In an amalgamated free product, $G = *_A G_i$, every element $g \in G$ is conjugate to an element in G_i for some i* (Serre, *Trees*, Section 1.3 Corollary 1 [28]*).*

Proposition 3.1.8. *The Klein bottle group is torsion-free.*

Proof. The fundamental group of the Klein bottle can be given as $\langle x, b \mid x^b = x^{-1}\rangle$ [21]. However, if we introduce $p = xb$ we see that x can be expressed as $x = pb^{-1}$ with $x^b = x^{-1}$ then derived as:

$$b^2 = p^2 = xbxb \implies b = xbx \implies bx^{-1}b^{-1} = x \implies bxb^{-1} = x^{-1}$$

So we can re-express the presentation as $\langle p, b \mid p^2 = b^2 \rangle$. This means $K = \langle p \rangle *_{p^2=b^2} \langle b \rangle$ and by an application of Lemma 3.1.7, we find K is torsion-free, because for all elements $g \in K$ we have $g = x^{-1}hx$ for $x, h \in \mathbb{Z}$. However, as $\mathbb{Z}$ is abelian, we have $g = h$, and h has infinite order. □

Proposition 3.1.9. *P is torsion-free.*

Proof. One way to show this is to apply Lemma 3.1.7 again. We know P is an amalgam of two Klein bottle groups, and the Klein bottle group is torsion free. □

Another interesting proof of the above is found in Section 1.4.1 of [8].

Definition 3.1.10. Given a surjective homomorphism $\varphi : G \to H$, a *lift* of an element $a \in H$ is an element $b \in G$ such that $\varphi(b) = a$.

Summary 3.1.11. Gardam, after expressing P as the amalgam of two Klein bottle groups, claims that the identified subgroups are normal in P, with quotient D_∞. He then chooses $z = abab$ as a lift to P of a generator of $[D_\infty, D_\infty]$. Let s and r, be the generators of D_∞. It can be shown that $[D_\infty, D_\infty]$ is generated by $(sr)^2$ by considering s as a shift along $\mathbb{Z}$ and r as a flip about a point on $\mathbb{Z}$.

Now, for x, y, z as defined in the presentation of P, we have that $\langle x, y, z \rangle \cong \mathbb{Z}^3$. This can be seen by showing x, y, z all commute. Note that the case for x^z is symmetric to the case for y^z. Firstly, we show $x^z = x$.

$(a^2)^{abab} = (abab)^{-1}a^3bab = b^{-1}a^{-2}b = b^{-2}a^2b^2 = (b^{-2}a)(ab^2) = (ab^2)(b^{-2}a) = x$

We get $x^y = x$ for free, as we showed above that $b^{-2}a^2b^2 = a^2$. So we have $\langle x, y, z \mid xy = yx, xz = zx, yz = zy \rangle$ which is isomorphic to $\mathbb{Z}^3$. Is is now clear that $\mathbb{Z}^3$ is the kernel of the mapping $\varphi : P \to \mathbb{Z}_2 \times \mathbb{Z}_2$ given explicitly by, $\varphi(a) \mapsto (-1, 1)$, $\varphi(b) \mapsto (1, -1)$ and $\varphi(ab) \mapsto (-1, -1)$.

Let $Q = \mathbb{Z}_2 \times \mathbb{Z}_2$, and set $z = abab$. Then the action of P on $\langle x, y, z \rangle$ by conjugation induces an action on Q, since $P/\mathbb{Z}^3 \cong Q$ as we recall from Example 3.1.2. We see $x^b = x^{-1}$ and $y^a = y^{-1}$ can be read off the presentation, $x^a = x$ and $y^b = y$ are also immediate. The rest are calculated below.

$x^{ab} = b^{-1}a^{-1}a^2ab = b^{-1}a^2b = x^b = x^{-1}$

$y^{ab} = a^{-1}b^{-1}b^2ab = b^{-1}a^{-1}b^2ab = b^{-1}y^{-1}b = b^{-1}b^{-2}b = y^{-1}$

$z^a z = bab(a^2b)ab = bab(xb)ab = bab(bx^{-1})ab = bab(ba^{-2})ab = babba^{-2}ab = b(ab^2a^{-1})b = b(b^{-2}a)a^{-1}b = 1$

$zz^b = b^{-1}ab(ab^2)abab = b^{-1}ab(b^{-2}a)abab = b^{-1}ab^{-1}a^2bab = b^{-1}a(b^{-1}a^2)bab = b^{-1}aa^{-2}b^{-1}bab = b^{-1}a^{-1}ab = 1$

$z^{ab} = b^{-1}a^{-1}ababab = abab = z$

3.2 Calculations and the Counterexample

We now present some of calculations required to verify that there is a non-trivial unit in $\mathbb{F}_2[P]$, following [9] closely. First we must discuss the normal form each element will be expressed in.

Recall $Q = \mathbb{Z}_2 \times \mathbb{Z}_2$. Define $\sigma : Q \to P$ explicitly by:

$$\sigma((1,1)) = 1 \qquad \sigma((-1,1)) = a$$
$$\sigma((1,-1)) = b \qquad \sigma((-1,-1)) = ab$$

the function σ is to act as the set-theoretic section of the previously defined function φ (defined in Summary 3.1.11). This means σ should return the preimage of φ given the image. However, this incurs an "error" which is dealt with by the cocycle $f : Q \times Q \to \mathbb{Z}^3$ defined as:

$$f(g,h) = \sigma(g)\sigma(h)\sigma(gh)^{-1}$$

This essentially tells us how to rearrange elements in P. Later we will present a table summarizing the inputs and outputs of this cocycle. At the bottom of page 4 in [9], Gardam shows how to push an a past a b resulting in the identity $ba = x^{-1}yz^{-1}ab$. We calculate below the easier identities showing how to push an a through x, y and z.

$$ax = aa^2 = a^2a = xa$$

$$ay = ab^2 = b^{-2}a = y^{-1}a$$

The second is obtained by considering the inverse of the relation $a^{-1}b^2 = b^{-2}a^{-1}$. To show $az = z^{-1}a$ requires a little more work. First we show the following:

$$b^{-1}a^2b = a^{-2}$$

$$b^{-1}a^2 = a^{-2}b^{-1}$$

$$ab^{-1}a^2 = a^{-1}b^{-1}$$

$$b^{-1}ab^{-1}a^2 = b^{-1}a^{-1}b^{-1}$$

With this in hand, we note that showing $az = z^{-1}a$ is equivalent to showing $aabab = b^{-1}a^{-1}b^{-1}a^{-1}a = b^{-1}a^{-1}b^{-1}$, which by the above working is the same as showing $aabab = b^{-1}ab^{-1}a^2$. The rest follows easily.

$$a^2bab = b^{-1}a^{-2}ab = b^{-1}ab^{-1}a^2$$

It follows similarly for b that:

$$bx = x^{-1}b \qquad by = yb \qquad bz = z^{-1}b$$

These identities save time when multiplying elements in $K[P]$. Repeated application of the above identity ($ba = x^{-1}yz^{-1}ab$), is enough to show that every element in $K[P]$ can be written in a normal form. Precisely stated, every $\beta \in K[P]$ can be written in the form $(\beta)_1 + (\beta)_a a + (\beta)_b b + (\beta)_{ab} ab$, where $(\beta)_c c$ is in $K[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$ for $c \in \{1, a, b, ab\}$. Recall again that $P/\mathbb{Z}^3 \cong Q$ by φ and so when we mod P out by $\langle x, y, z\rangle$, we are left with $1, a, b$ and ab. We can also see this by verifying every case as we do below.

Let $w \in P$ be a reduced word (for the precise meaning of "reduced word" see [27]) with n instances of a or a^{-1} and m instances of b or b^{-1} so that the length of w is $n+m$. Then, by repeated application of the above rules we obtain:

$$wx = x^{(-1)^m} w \qquad wy = y^{(-1)^n} w \qquad wz = z^{(-1)^{n+m}} w$$

These rules mean all instances of a^2, b^2 and $abab$ can be pulled to the left side of a string w in P. This means, without considering inverses and modulo $x^{\pm 1}, y^{\pm 1}, z^{\pm 1}$, there are only these cases to address.

$$\begin{aligned} aba &= ax^{-1}yz^{-1}ab &= y^{-1}zb \\ bab &= x^{-1}yz^{-1}ab^2 &= x^{-1}z^{-1}a \\ baba &= bax^{-1}yz^{-1}ab &= x^2z^{-1} \end{aligned}$$

However, any combination of inverses can be changed to the form $(\beta)_{ab}ab$:

$$a^{-1}b = x^{-1}ab \qquad b^{-1}a = y^{-1}ba = x^{-1}z^{-1}ab$$

$$ab^{-1} = yab \qquad ba^{-1} = x^{-1}yz^{-1}abx^{-1} = yz^{-1}ab$$

So it is now clear that by induction on the length of an element, w, of P, we can always reduce w to $(\beta)_1$, $(\beta)_a a$, $(\beta)_b b$ or $(\beta)_{ab}ab$ where $(\beta)_c$ for $c \in \{1, a, b, ab\}$ can be thought of as a monomial in variables $x^{\pm 1}, y^{\pm 1}$ and $z^{\pm 1}$. Therefore, after application of the above rules and factoring, any element β in $K[P]$ can be written as $\beta = (\beta)_1 + (\beta)_a a + (\beta)_b b + (\beta)_{ab}ab$. So this is another way to see what was clear from the observing the cosets of the factor group $P/\mathbb{Z}^3$.

These calculations also show how to get each entry in the table Gardam includes at the top of page five of [9], which we recreate below. This is the table for the cocycle defined at the beginning of this subsection.

$f(g,h)$	1	a	b	ab
1	1	1	1	1
a	1	x	1	x
b	1	$x^{-1}yz^{-1}$	y	$x^{-1}z^{-1}$
ab	1	$y^{-1}z$	y^{-1}	z

We can think about the table in familiar terms by recognizing that each entry shows how to rearrange a product of an element in the leftmost column and an element in the header row, such that the resultant element has a,b or ab as a suffix. So in the case of the a-row and the b-column, the product is ab which already has ab as a suffix, so the table reads 1. Below are some examples of multiplying elements of $K[P]$. Here we use the notation in [20] which says that if $f(x, y, z)$ is some Laurent polynomial in x, y and z then, for example, f_{xy} denotes $f(x^{-1}, y^{-1}, z)$, i.e., subscripts denote which variables have been flipped.

Example 3.2.1. let c and d be in $K[x^{\pm 1}, y^{\pm 1}, z^{\pm 1}]$. We wish to multiply cab and da and determine whether the suffix is $1, a, b$ or ab. Computing the product yields

$$cabda = cd_{xy}aba = cd_{xy}ax^{-1}yz^{-1}ab = x^{-1}zy^{-1}cd_{xy}a^2b = zy^{-1}cd_{xy}a$$

So we can say $zy^{-1}cd_{xy}$ has suffix a.

Example 3.2.2. Say that instead that we wish to compute the product $cbdab$, then:

$$cd_{xz}bab = cd_{xz}x^{-1}yz^{-1}abb = x^{-1}z^{z}cd_{xz}a$$

so $x^{-1}z^{-1}cd_{xz}$ has suffix a.

I emphasize that this notation is from Murray's 2021 paper [20] and Gardam denotes $x^{-1}z^{-1}cd_{xz}$ as $x^{-1}z^{-1}cd^{b}$ in [9]. Proceeding as shown in the above examples we can calculate the result of a general product of two elements written in the normal form. The equations given further down the page are almost exactly as they appear in Murray's paper (the original one in Gardam's paper is identical to Murray's but for the choice of notation). Each line is a member of one of the cosets, so that the entire product can be written

$$\alpha\alpha' = (\alpha\alpha')_1 + (\alpha\alpha')_a + (\alpha\alpha')_b + (\alpha\alpha')_{ab}$$

the factors, before expanding, are $\alpha\alpha' = (p+qa+rb+sab)(p'+q'a+r'b+s'ab)$.

$$\begin{array}{lclclclcl}
(\alpha'\alpha) & = & p'p & + & xq'q_{yz} & + & yr'r_{xz} & + & zs's_{xy} \\
(\alpha'\alpha)_a & = & p'q & + & q'p_{yz} & + & x^{-1}z^{-1}r's_{xy} & + & y^{-1}s'r_{xy} \\
(\alpha'\alpha)_b & = & p'r & + & xq's_{yz} & + & r'p_{xz} & + & y^{-1}zs'q_{xy} \\
(\alpha'\alpha)_{ab} & = & p's & + & q'r_{yz} & + & x^{-1}yz^{-1}r'q_{xz} & + & s'p_{xy}
\end{array}$$

Recall that to find a counterexample to the unit conjecture, we want to find a nontrivial unit, α. That is, we want $\alpha\alpha' = 1$ for some α', this requires that $(\alpha\alpha')_a$, $(\alpha\alpha')_b$ and $(\alpha\alpha')_{ab}$ vanish in the sum, and that $(\alpha\alpha')_1 = 1$. With this set-up, it is possible to verify Gardam's counterexample by hand. The nontrivial unit is given in Theorem A of [9]. For p, q, r, s as defined in Theorem A, the inverse is $x^{-1}p^{a} + x^{-1}qa + y^{-1}rb + z^{-1}s^{a}ab$ [11].

4 First-Order Logic and Model Theory

Many of the properties explored in Section 2 and Section 3 have expressions in first-order logic (FOL). As we will see, it is also possible for a property, given some language, to not have a first-order expression.

In a seminar talk on groups and first-order logic, André Nies provided three reasons to care about the intersection of these two areas.

- First-order properties of a group, G, can be verified within G.
- First-order logic has a "toolbox", e.g., compactness theorem.
- The first-order theory of a group is an invariant, that is, preserved under isomorphism.

Definition 4.0.1. The language of groups, $\mathcal{G}_{\mathcal{L}}$, contains the following: A constant symbol, denoted 1_G or simply 1, which is the identity. A unary function, denoted (with input g) as g^{-1}, and a binary multiplication function which can be represented as $g_1 \cdot g_2$. Sometimes the function symbol is omitted and we write $g_1 g_2$

For a comprehensive treatment of languages, models and related preliminary material, please see section 1.3 of Chang and Keisler's *Model Theory* 3rd edition (1990) [5].

4.1 Expressing Properties in First-Order Logic

We begin by showing that first-order logic cannot describe every property. More precisely we show there is a formal theory in the first-order language of groups containing a sentence which can't be expressed. We first state the compactness theorem for first-order logic.

Theorem 4.1.1. Compactness. *let Σ be a set of first-order sentences over some first-order language $\mathcal{L}$. If every finite subset of Σ has a model, then Σ has a model.*

The following proposition shows that we can construct a formal theory in the first-order language of groups

Proposition 4.1.2. *The property of being a torsion group, is not expressible in the language of groups.*

Proof. Let $\mathcal{L}_{\mathcal{G}}$ be the language of groups and let T_t be a first-order theory of groups with torsion. We add a constant symbol c to our language, so we have $\mathcal{L}_{\mathcal{G}} \cup \{c\}$ and we define a new theory $T = T_t \cup \{c^n \neq 1_G \mid 1 \leq n\}$, ($c^n$ is shorthand for $c \cdot c$, n times).

We can't have a model for T, because any model M of T is a group with torsion, but the added axioms say c^M must have infinite order. So by the compactness theorem, any finite subset $T_1 \subseteq T$ also cannot have a model. However, $T_1 \cup T_t$ clearly has a model, take $m = \max\{n \mid (c^n \neq 1_g) \in T_1\}$ and consider the largest group satisfying the presentation $\langle x \mid x^{m+1} = 1_G \rangle$, this is a group with torsion which satisfies the formulas in T_1, which is a contradiction.

□

We now know that some structures and their properties cannot be expressed in first-order logic. In the Logic Blog 2021, André Nies considers FO expressions for the properties of groups and group rings defined in the previous section.

Expression 4.1.3. We express one-sided orderable (Definition 2.1.5) by using Conrad's criterion, Theorem 2.2.10. G is one-sided orderable if and only if for each $n \geq 1$ and each non-identity elements $x_0, \dots, x_{n-1}$ in G, there is a tuple $\sigma \in \{-1, 1\}^n$ such that e is not in the semigroup generated by the $x_i^{\sigma_i}$.

For each n and each family of semigroup terms $\bar{t} = (t_\sigma)_{\sigma \in \{-1,1\}^n}$, let $\phi_{\bar{t}}$ be the sentence

$$\forall x_0 \neq e, \dots, x_{n-1} \neq e \bigvee_{\sigma \in \{-1,1\}^n} t_\sigma(x_0^{\sigma_0}, \dots, x_{n-1}^{\sigma_{n-1}}) \neq e.$$

We claim that Conrad's criterion is expressed by satisfying all the $\phi_{\bar{t}}$.

If the criterion holds then clearly each of the sentences holds.

Now suppose that the criterion fails for G. So there is n and there are $g_1, \dots, g_n \in G - \{e\}$ such that for each $\sigma \in \{-1, 1\}^n$, the semigroup generated by the $g_i^{\sigma_i}$ contains e. So for each σ we can choose a semigroup term t_σ such that $t_\sigma(g_0^{\sigma_0}, \dots, g_{n-1}^{\sigma_{n-1}}) = e$. Then G fails $\phi_{\bar{t}}$ where $\bar{t} = (t_\sigma)_{\sigma \in \{-1,1\}^n}$ [22].

Expression 4.1.4. The property of being diffuse given in Definition 2.1.8 is expressed by the following set of sentences, where $x_1, \dots, x_n$ are variables and the set C is considered as $\{x_1, \dots, x_n\}$

$$\forall x_1, \dots, x_n \bigvee_{i \leq n} \forall y (\bigwedge_k y x_i \neq x_k \vee \bigwedge_k y^{-1} x_i \neq x_k) [22]$$

Inside the brackets, we iterate over the set with "and", and outside of the brackets with "or". Note that $yx_i \neq x_k$, for each k, is equivalent to saying $yx_i \notin C$.

Expression 4.1.5. The unique product property given in Definition 2.1.1 can be expressed as follows.

$$\forall x_1 \dots x_n \forall y_1 \dots y_n \bigvee_{i,k \leq n} (\bigwedge_{r,s \leq n} (x_i y_k = x_r y_s \rightarrow x_i = x_r))[22]$$

Note that $x_i = x_r$ gives $y_k = y_s$ by cancellation.

Expression 4.1.6. The *two unique products property* given in Definition 2.1.2 is:

$$\forall x_1, \dots, x_n \forall y_1, \dots, y_n$$

$$\bigvee_{i,j \leq n} (\bigwedge_{r,s \leq n} x_i y_j = x_r y_s \rightarrow x_i = x_r) \wedge \bigvee_{p \neq i, q \neq j \leq n} (\bigwedge_{r,s \leq n} x_p y_q = x_r y_s \rightarrow x_p = x_r)$$

Additionally, the constraint that the sets $\{x_1, \dots, x_n\}$ and $\{y_1, \dots, y_n\}$ should not both be singletons can be expressed as:

$$\forall x_1 \dots x_n \forall y_1 \dots y_n \bigwedge_{i,p \leq n} (x_i \neq x_p \vee y_i \neq y_p)$$

Like in the case of the Expression 4.1.3, Expressions 4.1.4, 4.1.5 and 4.1.6 all require that we use a family of first-order sentences. One sentence for each size n of the sets.

Expression 4.1.7. The goal is to express that $K[G]$ has no zero divisors. Nies begins by defining a function to code a partial multiplication table T, for some group G. Where $r, s, l \in \mathbb{P}$ and $l \leq rs$.

$$T = \{1, \dots, r\} \times \{1, \dots, s\} \rightarrow \{1, \dots, l\}$$

For a table T, he then defines the following system of equalities and inequalities:

$$E_T = \bigvee_{i \leq r} \alpha_i \neq 0 \wedge \bigvee_{k \leq s} \beta_k \neq 0 \wedge \bigwedge_{u \leq l} [\sum_{T(i,k)=u} \alpha_i \beta_k = 0][22]$$

Here, α_i,β_k are taken to be members of an algebraically closed field (note that any field can be extended to an algebraically closed field), K. The above is really saying that for pairs of elements in K, not all 0, we set the products equal to 0 which correspond to the $(i, k)^{th}$ position in the table T. Together, this says that if the table is realised by elements $g_1, \dots, g_r$, $h_1, \dots, h_s$, $v_1, \dots, v_l$

with each g_i and h_k being pairwise distinct, then assigning the coefficients α_i and β_k to g_i and h_k respectively yields:

$$(\Sigma\alpha_i g_i)(\Sigma\beta_k h_k) = \Sigma(\alpha_i\beta_k)(g_i h_k) = 0$$

Meaning $\Sigma\alpha_i g_i$ and $\Sigma\beta_k h_k$ are zero-divisors. However, the following sentence in $\mathcal{L}_{\mathcal{G}}$:

$$\phi_T = \forall x_1, \ldots, x_r \forall y_1, \ldots y_s \forall z_1, \ldots, z_l$$

$$\bigvee_{i<j\leq r} x_i = x_j \vee \bigvee_{i<j\leq s} y_i = y_j \vee \bigvee_{i\leq r\wedge k\leq s} z_{T(i,k)} \neq x_i y_k,$$

[22] states that T cannot be realised by a column of pairwise distinct elements and a row of pairwise distinct elements, and hence that the product $(\Sigma\alpha_i g_i)(\Sigma\beta_k h_k)$ cannot occur, which means that $K[G]$ has no zero divisors. Therefore both expressions taken together say that that $K[G]$ has no zero divisors.

We need to argue that the above set of sentences is actually computable. Firstly, the theory of algebraically closed fields, ACF, is decidable (Definition 4.2.1). This argument follows Definition 2.2.7 to Corollary 2.2.9 in Marker's *Model Theory: An Introduction* (2002)[19]. The argument shows that the theory of Algebraically closed fields is decidable by applying the Completeness Theorem and using that ACF is a recursively enumerable language. Therefore, because ACF is decidable, the set of sentences above is computable

Example 4.1.8. An example of the previous statement "$K[G]$ has no zero divisors" for some torsion free group G, is given by the following partial multiplication table T.

$\cdot$	g_0	g_1
h_0	$h_0 g_0$	$h_0 g_1$
h_1	$h_1 g_0$	$h_1 g_1$

We are given that $(g_0 + g_1)(h_0 + h_1) = 0$ over $\mathbb{F}_2$. Like in the general expression, we will find that T is impossible and thus the given zero divisor cannot occur. Recall that g_1 and g_0 are distinct, and likewise for h_0 and h_1, so for $g_0h_0 + g_0h_1 + g_1h_0 + g_1h_1 = 0$ we therefore must have $g_1h_1 = g_0h_0$ and $g_0h_1 = g_1h_0$.

This means that $g_1 = g_0h_1h_0^{-1} = g_0h_0h_1^{-1}$ and so $h_1h_0^{-1} = h_0h_1^{-1}$ but we can see that $(h_1h_0^{-1})^{-1} = h_0h_1^{-1}$ meaning the element $h_1h_0^{-1}$ is self-inverse, which contradicts that h_0 and h_1 are distinct.

4.2 Boolean Satisfiability

In this section we examine how Giles Gardam transformed the problem of finding nontrivial units in a group ring into a Boolean satisfiability problem. This is an effective approach because algorithms for solving Boolean satisfiability problems (SAT solvers) are practically very good, in part, due to SAT solver competitions [15].

We first need to categorize decision problems, which are problems that, on some input, output either YES or NO.

Definition 4.2.1. A decision problem can fall into the following categories, given input x and a set A:

> **Decidable:** There is a deterministic algorithm which outputs YES if $x \in A$ and outputs NO if $x \notin A$.
>
> **Semidecidable:** There is a deterministic algorithm, which outputs YES if $x \in A$ and outputs NO, or runs forever if $x \notin A$.
>
> **Undecidable:** There is no deterministic algorithm outputting YES or NO, whether $x \in A$ or $x \notin A$.

There are semidecidable and undecidable problems in Group theory.

Theorem 4.2.2. *(Novikov–Boone) The word problem for groups is undecidable.*

Proof. Novikov's proof is in [23] and Boone's is in [1] □

Gardam says that finding non-trivial units in a group ring is semidecidable "modulo the word problem", meaning we can find non-trivial units, except for the issues posed by determining if two arbitrary group elements are the same. He notes further that restricting the problem makes it more plausible.

As we know from Section 3, Gardam found his nontrivial unit by considering the field $\mathbb{F}_2$, he further observes that, more generally, given two subsets $A, B \subset G$, the problem of determining if they support a non-trivial solution to $\alpha\beta = 1$ over $\mathbb{F}_q$ is in NP. Because SAT is NP-Complete (Cook-Levin theorem), it must be possible to reduce the problem to a satisfiability problem. For an explanation of NP-Completeness and The Cook-Levin Theorem see section 7.4 of Sipser's *Introduction to the Theory of Computation* [29].

Definition 4.2.3. A Boolean formula is in *conjunctive normal form* (CNF) if it has the following structure.

$$(x_i \vee \ldots \vee x_j) \wedge \ldots \wedge (x_r \vee \ldots \vee x_s)$$

Where i, j, r, s are in some indexing set I.

Each variable x_i or its negation $\overline{x_i}$ is called a literal, and each disjunction of literals in a CNF formula is called a clause.

Once a problem is phrased as a satisfiability problem, it can be transformed into CNF via a Tseytin transformation (shown below). This is done because almost all SAT solvers work on CNF formulas.

Example 4.2.4. A Tseytin transformation runs the following algorithm on a logical formula φ:

1. Set new auxiliary variables equivalent to each subformula (must include a connective) of φ
2. Write a new formula ψ, which is the conjunction of each equivalence from step 1, and also the formula which asserts φ. So that ψ says "ϕ AND (first substitution) AND (second substitution)... is true.
3. Use logical laws to convert each "if and only if" statement into CNF.

We perform a partial transformation on the formula $p \rightarrow (s \vee q)$.

1. $x_1 \leftrightarrow s \vee q$, $x_2 \leftrightarrow (p \rightarrow (s \vee q))$
2. $\psi = x_2 \wedge (x_1 \leftrightarrow s \vee q) \wedge (x_2 \leftrightarrow (p \rightarrow (s \vee q)))$
3.

$$\begin{aligned} & x_1 \leftrightarrow s \vee q \\ \iff & (x_1 \rightarrow s \vee q) \wedge (s \vee q \rightarrow x_1) \\ \iff & (\neg x_1 \vee s \vee q) \wedge ((\neg s \wedge \neg q) \vee x_1) \\ \iff & (\neg x_1 \vee s \vee q) \wedge (\neg s \vee x_1) \wedge (\neg q \vee x_1) \end{aligned}$$

Here we have only transformed the second clause in ψ, but the third follows similarly from applying logical laws.

To convert the problem of finding non-trivial units in a group ring into Boolean logic we need to write that there are non-trivial elements α and β in $K[G]$ with product 1, that is, encode $\alpha\beta = 1$ as a Boolean expression. Note that $kg \in K[G]$ is 0 if and only if $k = 0$. The ball of radius n over words on G, is denoted $B(n)$, and is the set $\{x \in G \mid |x| \leq n\}$. Elements α and β of $K[G]$ are expressed as follows:

$$\alpha = \sum_{g \in B(n)} a_g g \qquad \beta = \sum_{g \in B(n)} b_g g$$

Where a_g and b_g are field elements. The non-triviality of the elements is expressed like:

$$a_1 \wedge \bigvee_{g \in B(n) \setminus \{1\}} a_g$$

The clause "a_1" asserts that the field element a_1 is non-zero, since each Boolean variable can have value 0 or 1, recall $\mathbb{F}_2 = (\{0,1\}, +, \cdot)$. The disjunction is interpreted similarly. Here, a_1 is being thought of as the identity element of the field.

Next, we set up a quadratic system of equations, the solution of which will yield the desired α and β. To convert to CNF, we require that the variable $x_{g,h}$ is introduced. Define $x_{g,h} = a_g \cdot b_h$. Observe that $a_g \cdot b_h \in \mathbb{F}_2$ is equivalent to $a_g \wedge b_h$ where a_g and b_h are Boolean variables. This results in the following expression:

$$\phi_\times = (\overline{x_{g,h}} \vee a_g) \wedge (\overline{x_{g,h}} \vee b_h) \wedge (\overline{a_g} \vee \overline{b_h} \vee x_{g,h})$$

As Gardam notes, the above holds if and only if the following holds:

$$(x_{g,h} \to a_g) \wedge (x_{g,h} \to b_h) \wedge ((a_g \wedge b_h) \to x_{g,h})$$

So we see that $(x_{g,h} \leftrightarrow a_g \wedge b_h) \leftrightarrow \phi_\times$.

Next, we wish to express each equation in the system. For this we need the coefficient of a given group element to sum to 0, unless the group element is 1, in which case it should sum to 1.

This is achieved by setting $(a_g \wedge b_h) = 1$ when g and h are trivial, setting $(a_g \wedge b_h) = 1$ when $h = g^{-1}$ and setting $(a_g \wedge b_h) = 0$ when $gh = w$ for some $w \in B(n)$ for some n. This asserts each equation in the system.

$$\sum_{h,g \in B(n)} a_g b_h gh$$

However, this still needs to be put into CNF. Gardam provides the following example of how this is done.

Example 4.2.5. The equation $x+y+z=0$ (in $\mathbb{F}_2$) takes the following form as a Boolean expression:

$$(\overline{x} \vee \overline{y} \vee \overline{z}) \wedge (\overline{x} \vee y \vee z) \wedge (x \vee \overline{y} \vee z) \wedge (x \vee y \vee \overline{z})$$

Each clause is the negation of a possible non-solution to the above equation. For example, $x=1$, $y=1$ and $z=1$, or $(x \wedge y \wedge z)$ is not a solution to the equation, so is ruled out by the constraint $(\overline{x} \vee \overline{y} \vee \overline{z})$ which is its negation.

4.3 Generalizing Beyond Characteristic 2

Less than two months after Gardam's paper was posted, Alan Murray extended the counterexample over $\mathbb{F}_2$ to a class of counterexamples over fields, $\mathbb{F}_p$, of every prime characteristic [20]. This class of counterexamples can unfortunately not be generalized to characteristic 0, as one might hope.

Gardam pointed out in his talk on Kaplansky's conjectures in September 2021, that if we had counterexamples to any of Kaplansky's conjectures (of uniformly bounded support) for all finite fields, then we would have counterexamples for fields of characteristic 0. The next proposition and its proof are informed by Gardam's explanation in the aforementioned talk [10].

Proposition 4.3.1. *A uniformly bounded set of counterexamples $\{\phi_i \mid i \in I\}$ falsifying the unit conjecture in all finite fields would yield counterexamples for fields of characteristic 0.*

Proof. Note that we can express the unit conjecture in first-order logic by a formula similar to Expression 4.1.7, see [22]. Because the family of counterexamples $\{\phi_i \mid i \in I\}$ is uniformly bounded, we can find M so that $|\phi_i| \leq M$ for all $i \in I$ and so we know there is a counterexample with finite support falsifying the unit conjecture for each finite field. We know that the algebraic closure of $\mathbb{F}_p$ can be given as $\overline{\mathbb{F}_p} = \bigcup_k \mathbb{F}_{p^k}$. So then we can find a counterexample for algebraically closed fields of arbitrarily large p. Therefore, by the Lefschetz principle (which is a consequence of compactness, Theorem 4.1.1), we have that some ϕ_i with $i \in I$ holds in every algebraically closed field of characteristic 0. This argument could have also been made with ultrafilters (which are used in 4.3.3) like Gardam does in [10]. □

Before proceeding, we need the Fundamental Theorem of Ultraproducts (FTU), and also sometimes "Łoś's Theorem". Note that "Łoś" is a Polish name and can be pronounced similar to *wash*. We will also only use the third part of this theorem so will denote it FTU3. For the background definitions, please refer to chapter 4 of Chang and Keisler's Model Theory (1990) [5].

Theorem 4.3.2. FTU3 *Let D be a filter and $\mathcal{B}$ be the ultraproduct $\prod_D \mathcal{U}_i$, with I as the index set. Then for any sentence φ in some language $\mathcal{L}$:*

$$\mathcal{B} \models \varphi \text{ if and only if } \{i \in I \mid \mathcal{U}_i \vdash \varphi\} \in D$$

which says that φ holds in the ultraproduct if and only if it holds in *most* of the components.

Theorem 4.3.3. *If $\mathcal{U}_i$ is a collection of fields, with only finitely many having characteristic p for each prime, then the ultraproduct $\prod_D \mathcal{U}_i$ has characteristic 0.*

Proof. By construction, some member of the ultraproduct, $\mathcal{U}_j$, does not have characteristic p for some prime p. This means $p \neq 0$ in $\mathcal{U}_i$ and since $\mathcal{U}_i$ is a field, p has a multiplicative inverse. So we can say $\phi_p = \exists x(px - 1 = 0)$, holds in $\mathcal{U}_i$.

Again, by construction, ϕ_p holds for all but finitely many members of $\mathcal{U}_i$, meaning $\{i \mid \mathcal{U}_i \vdash \phi_p\} \subseteq I$ is confinite. We now use the result (without proof) that an ultrafilter contains a filter over a cofinite index (called a Fréchet filter) and thus we can apply FTU3 to find that $\prod_D \mathcal{U}_i \vdash \phi_p$ and so, since the argument can be repeated for all p, we see $\prod_D \mathcal{U}_i$ must have characteristic 0. □

Let P be the set of all primes. Using the above theorem we can claim that if K is an ultraproduct of $\mathbb{F}_p$ for all $p \in P$, then K has characteristic 0 since for each characteristic, a field with that characteristic appears only once in the ultraproduct.

Murray's counterexamples as given in Theorem 3 of [20], depend on the prime p and thus the support grows as p grows. Hence, if we had a counterexample for characteristic 0, as described in the paper, it would not have finite support and so could not be in the group ring. The unit conjecture for fields of characteristic 0 remains open.

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