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ON TOPOLOGIES GENERATED BY MOISIL RESEMBLANCE RELATIONS

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In [8] and [9] Moisil has introduced the resemblance relations. Following [9] we associate to every resemblance relation an extensive operator which commutes with arbitrary unions of sets. We are leading to consider spaces endowed with such closure operators; we shall call these spaces *total Čech spaces* (*TC-spaces*).

TC-spaces are in one-to-one, onto correspondence with reflexive relations. TC-spaces generated by transitive relations are in one-to-one, onto correspondence with the total topological spaces of W. Hartnett (which are called *total Kuratowski spaces*, *TK-spaces*).

We study the category of TC-spaces and its full subcategory determined by TK-spaces. Both categories are Cartesian closed, but they are not elementary toposes.

1. Resemblance relations

Let X be a set and $S \subseteq X^3$. We denote by $S(a, b, c)$ the relation "$(a, b, c) \in S$". Following Moisil [9, p. 15] S is a *resemblance relation* on X if the following three axioms hold:

(I) For any $a, c \in X$, $S(a, a, c)$.

(II) For any $a, b \in X$, $S(a, b, b)$.

(III) For any $a, x, y, z \in X$, if $S(a, x, y)$ and $S(a, y, z)$, then $S(a, x, z)$.

A resemblance space is a set X together with a resemblance relation S on X. The axioms (I)–(III) are derived from the following interpretation of the relation S: $S(a, b, c)$ means "a resembles to b at least as it resembles to c".

We give three examples of resemblance spaces.

Example 1 (G.C. Moisil). Let C be an oriented circle. For the points a, b, c on C we have $S(a, b, c)$ if the points a and c are joined only if a and b, respectively, b and c, are joined (on the circle).

Example 2. Let $(X, \mathscr{T})$ be a topological space. For $a, b, c \in X$ we have $S(a, b, c)$ if any neighborhood of a which contains c, contains also b.

Example 3. Let (X, d) be a metric space. Define $S(a, b, c)$ if $d(a, b) \leq d(a, c)$.

Let (X, S) be a resemblance space. For every $a \in X$ we define an equivalence relation P_a on X, as follows: $P_a(b, c)$ if $S(a, b, c)$ and $S(a, c, b)$. The relation P_a can be interpreted (following [9]) as the *indifference relation* associated to S and a. In the relation S we can fix the first argument, $S_a(x, y) = S(a, x, y)$. We obtain a preorder relation on X, with a as first element; moreover, S_a is compatible with P_a.

Conversely, if X is a set together with a family of preorder relations $\{<_a\}_{a\in X}$ such that any $<_a$ has a as first element, then the relation

$$S(a, b, c) \quad \text{if} \quad b <_a c$$

is a resemblance relation on X. This correspondence is one-to-one and onto.

Let a and r be in X. The set $V_a^r = \{x \mid S(a, x, r)\}$ is called the *resemblance neighborhood of a* (of degree r).

Lemma 1. *Let (X, S) be a resemblance space. The following three conditions are equivalent:*

(i) $S(a, b, c)$,

(ii) *For any $r \in X$, $c \in V_a^r$ implies $b \in V_a^r$,*

(iii) $V_a^b \subseteq V_a^c$.

Proof. Let us suppose that $S(a, b, c)$. If $r \in X$ and $c \in V_a^r$, then $S(a, c, r)$; it follows that $S(a, b, r)$, that is $b \in V_a^r$.

We have $c \in V_a^c$; hence $b \in V_a^c$, that is $S(a, b, c)$. Now $x \in V_a^b$ means that $S(a, x, b)$ hence $S(a, x, c)$, that is $x \in V_a^c$.

Because $b \in V_a^b$ we obtain $b \in V_a^c$, that is $S(a, b, c)$.

Corollary 1. *For any $a, b \in X$, V_a^b is the smallest resemblance neighborhood of a which contains b; V_a^a is the smallest resemblance neighborhood of a.*

Proof. Let a, b, r be in X. If $b \in V_a^r$, then $S(a, b, r)$, therefore $V_a^b \subseteq V_a^r$. Moreover, in view of $S(a, a, b)$ it follows that $V_a^a \subseteq V_a^b$.

We denote by 2^X the power set of X.

Lemma 2. *Let (X, S) be a resemblance space. The operator $\varphi_S : 2^X \to 2^X$ defined by $\varphi_S(A) = \{x \mid x \in X, V_x^x \cap A \neq \emptyset\}$ has the following two properties:*

(1) $A \subseteq \varphi_S(A)$, *for every $A \in 2^X$,*

(2) *for any family $\{A_i\}_{i\in I}$ of subsets of X,*

$$\varphi_S\Big(\bigcup_{i\in I} A_i\Big) = \bigcup_{i\in I} \varphi_S(A_i).$$

It is also known that such an operator is monotone and $\varphi_S(\emptyset) = \emptyset$.

2. Total Čech spaces

A Čech space is a couple (X, φ), $\varphi: 2^X \to 2^X$, where φ is extensive and monotone [3]. Lemma 2 suggests to consider Čech spaces endowed with the following "total property": The family of open sets is closed under arbitrary intersections (property credited to Hartnett in [7]). This property is equivalent to the property (2) in Lemma 2.

A *total Čech space* (*TC-space*) is a couple (X, φ), $\varphi: 2^X \to 2^X$, such that the following two conditions hold:

(1) $A \subseteq \varphi(A)$, for any $A \subseteq X$,

(2) for every family $\{A_i\}_{i \in I}$ of subsets of X,

$$\varphi\left(\bigcup_{i \in I} A_i\right) = \bigcup_{i \in I} \varphi(A_i).$$

Note that any function $\varphi: 2^X \to 2^X$ having the property (2) is uniquely determined by a function $\varphi: X \to 2^X$, where $\varphi(x) = \varphi(\{x\})$, for every $x \in X$, and $\varphi(A) = \bigcup_{a \in A} \varphi(a)$, for any $A \subseteq X$. Property (1) is equivalent to the condition "$x \in \varphi(x)$, for every $x \in X$". Moreover, we have:

Theorem 1. *There is a one-to-one, onto correspondence between TC-spaces and reflexive relations.*

Proof. To every TC-space (X, φ) we associate the reflexive relation $(X, R\varphi)$, where $xR\varphi y$ iff $y \in \varphi(x)$. Conversely, to the reflexive relation (X, R) we associate the TC-space (X, φ_R), where $\varphi_R(A) = \{y \mid xRy, \text{ for some } x \in A\}$. One can easily verify that $R\varphi_R = R$ and $\varphi_{R\varphi} = \varphi$.

An A-space is a Čech space (X, φ), where φ is finitely additive [3]. So, any TC-space is an A-space but the converse fails; any A-space is a Čech space and the converse, also, fails.

In the study of A-space one works with the following concepts. A set $A \subseteq X$ is *closed* if $\varphi(A) = A$; the set A is *open* if $X \setminus A$ is closed; the set V is a (*topological*) *neighborhood* of $a \in X$ if $a \notin \varphi(X \setminus V)$. We denote by $\mathcal{V}(a)$ the set of all neighborhoods of a.

We display some of the properties of TC-spaces which are derived from the properties of A-spaces [3].

Proposition 1. *Let (X, φ) be an A-space. The following statements hold*:

(1) *X and ϕ are both closed and open*;

(2) *the family of all open (closed) sets is closed under arbitrary unions (intersections)*;

(3) *if $a \in X$ and $V \in \mathcal{V}(a)$, then $a \in V$*;

(4) *if $V \in \mathcal{V}(a)$ and $V \subseteq W \subseteq X$, then $W \in \mathcal{V}(a)$*;

(5) *if $a \in X$ and $A \subseteq X$, then $a \in \varphi(A)$ iff $V \cap A \neq \emptyset$, for any $V \in \mathcal{V}(a)$*;

(6) *a set $A \subseteq X$ is open iff $A \in \mathcal{V}(a)$, for every $a \in A$.*

In a TC-space if $\{V_i\}_{i\in I}$ is a family of neighborhoods of an element $a \in X$, then so is $\bigcap_{i\in I} V_i$. We denote by $W_a = \bigcap_{V\in\mathscr{V}(a)} V$; thus, $W_a \in \mathscr{V}(a)$. Moreover, $V \in \mathscr{V}(a)$ iff $W_a \subseteq V$.

From Proposition 1(5), in an A-space $\varphi(A) = \{a \mid A \cap V \neq \emptyset$, for every $V \in \mathscr{V}(a)\}$. In particular, in a TC-space, $\varphi(A) = \{a \mid W_a \cap A \neq \emptyset\}$.

Proposition 2. *Let (X, φ) be a TC-space. The following two statements hold*:

(1) $W_a = \{x \mid x \in X, a \in \varphi(x)\}$,

(2) $\varphi(x) = \{a \mid a \in X, x \in W_a\}$.

Proof. The relation $x \in W_a$ is equivalent to $x \in V$, for every $V \in \mathscr{V}(a)$, that is, for any neighborhood V of a, $V \cap \{x\} \neq \emptyset$; the last relation is equivalent to $a \in \varphi(x)$.

Corollary 2. *A TC-space is uniquely determined by a family of subsets of X, $\{W_a\}_{a\in X}$, with the property $a \in W_a$. The set W_a is the smallest neighborhood of the element a.*

If $\varphi = \varphi_R$ where R is a reflexive relation, then $x \in W_a$ iff xRa, for all $x, a \in X$.

In a TC-space the family of open (closed) sets is closed under arbitrary intersection (unions). Hence, the family of open (closed) sets forms a complete Brouwerian lattice (i.e. a complete lattice for which $b \cap (\bigcup_{i\in I} a_i) = \bigcup_{i\in I} (b \cap a_i)$); in particular, these families form Heyting lattices (where $a \rightarrow B = \bigcup \{x \mid a \cap x \subseteq b\}$).

In a TC-space (X, φ) we define the following resemblance relation $S_\varphi : S_\varphi(a, b, c)$ if for every $V \in \mathscr{V}(a)$, $c \in V$ implies $b \in V$. Let us note that $S\varphi(a, b, c)$ iff $b = c$ or $b \in W_a$. Thus, to any TC-space (X, φ) we associate a resemblance space $(X, S\varphi)$, and to any resemblance space (X, S) we associate a TC-space (X, φ_S). We ask whether $(X, S) = (X, S\varphi_S)$ and $(X, \varphi) = (X, \varphi_{S\varphi})$?

Theorem 2. *Any TC-space (X, φ) is of the form (X, φ_S), where (X, S) is a resemblance space.*

Proof. From the definition of φ_S it follows that the space (X, φ_S) is uniquely determined by the family $W_a = V_a^a$, $a \in X$.

Obviously, xRy iff $x \in V_y^y$ iff $S(y, x, y)$. Now, in the space $(X, S\varphi)$, $V_a^r = \{x \mid S\varphi(a, x, r)\} = W_a \cup \{r\}$. In the space $(X, \varphi_{S\varphi})$ the smallest neighborhood of a is just the smallest resemblance neighborhood of a $W_a = V_a^a$. Therefore, $\varphi_{S\varphi} = \varphi$.

Proposition 3. *For any resemblance relation S, $S\varphi_S \subseteq S$; $S\varphi_S = S$ iff for all a, b, c, the relation $S(A, b, c)$ and $b \neq c$ imply $S(a, b, a)$.*

Proof. If $S\varphi_S$ (a, b, c), then for any $V \subseteq X$ with $a \notin \varphi_S(X \setminus V)$ and $c \in V$ imply $b \in V$. But, in view of Proposition 1(4) $a \notin \varphi_S(X \setminus V_a^c)$ and therefore $S(a, b, c)$.

The equality $S = S\varphi_S$ holds iff for any a, $c \in X$ the equalities $W_a \cup \{c\} = V_a^c$ hold. Then $S = S\varphi_S$ is equivalent to $V_a^c = V_a^a \cup \{c\}$ for all a, c, X, i.e., for all a, b, $c \in X$, $b \in V_a^c$ and $b \neq c$ imply $b \in V_a^a$.

The condition in Proposition 3 fails in general. For example, let us take $X = \{1, 2, 3\}$, $S = \{(1, 1, 1), (1, 2, 2), (1, 3, 3), (1, 1, 2), (1, 1, 3), (2, 2, 2), (2, 2, 1), (2, 2, 3), (2, 1, 1), (2, 3, 3), (3, 3, 3), (3, 3, 1), (3, 3, 2), (3, 1, 1), (3, 2, 2), (1, 2, 3)\}$.

A topological space (in the sense of Kuratowski) is called *total* (following W. Hartnett, cited by [7]) if the family of closed sets is closed under arbitrary unions. Such a space will be called *total Kuratowski space* (*TK-space*). A Kuratowski topological space is total iff the corresponding closure operator commutes with arbitrary unions of sets; thus, any TK-space is a TC-space. Moreover, we have:

Theorem 3. *Let R be a reflexive relation on X. Then, (X, φ_R) is a TK-space iff R is transitive.*

Proof. If R is transitive, $A \subseteq X$ and $x \in \varphi_R(\varphi_R(A))$, then there exist $z \in A$ and $y \in \varphi_R(z)$ such that $x \in \varphi_R(y)$. It results that zRy and yRx; therefore by transitivity, $x \in \varphi_R(z)$, with $z \in A$. The last relation shows that $x \in \varphi_R(A)$.

Conversely, let (X, φ_R) be a TK-space. If xRy and yRz, then $z \in \varphi_R(\varphi_R(x))$. In view of the relation $\varphi_R(\varphi_R(x)) = \varphi_R(x)$ we deduce xRz.

We note that to every TC-space (X, φ) one can associate a TK-space $(X, \mathscr{C})$, where $\mathscr{C}(A) = \bigcup_{n \geq 0} \varphi^n(A)$. $R_{\mathscr{C}}$ is just the transitive closure of $R\varphi$.

In [6] one studied TC-space induced by various types of reflexive relations (tolerance, antisymmetric and order relations).

3. Categorical properties of TC-spaces

Let (X, φ) and (Y, ψ) be two TC-spaces. The function $f:(X, \varphi) \to (Y, \psi)$ is called a *morphism of TC-spaces* if f is a function from X to Y and for any $A \subseteq X$, $f(\varphi(A)) \subseteq \psi(f(A))$. The last condition is equivalent to the condition: For every $x \in X$, $f(\varphi(x)) \subseteq \psi(f(x))$.

Theorem 4. *The category of TC-spaces* (TopTC) *is isomorphic with the category of reflexive relations.*

Proof. Let us denote by $R\varphi$ and $R\psi$ the reflexive relations associated with the operators φ and ψ, respectively. To prove the theorem it suffices to observe that a function $f: X \to Y$ is a morphism of TC-spaces iff for any x, $y \in X$, $xR\varphi y$ implies $f(x)R\psi f(y)$.

For details concerning the categories of preorder (order) relations see [5] and [10].

We display the main properties of the category TopTC and of the forgetful functor U:TopTC$\to$Set.

The family $f_i:(X,\varphi)\to(X_i,\varphi_i)$ of morphisms in TopTC is monic (i.e. for any i, $f_iu=f_iv$ implies $u=v$) iff the condition: $f_i(x)=f_i(y)$, for any i, implies $x=y$. The family (f_i) is initial (i.e. a function $f:(Y,\psi)\to(X,\varphi)$ is a morphism in TopTC iff for any i, f_if is a morphism in TopTC (see [1])) iff for any $A\subseteq X$,

$$\varphi(A)=\bigcup_{x\in A}\bigcap_i f_i^{-1}(\varphi_i(f_i(x))).$$

The family $f_i:(X_i,\varphi_i)\to(X,\varphi)$ of morphisms in TopTC is epic iff $X=\bigcup_i f_i(X)$. The family (f_i) is terminal iff for any $A\subseteq X$, $\varphi(A)=A\cup\bigcup_i f_i(\varphi_i(f_i^{-1}(A)))$.

The functor U has right (left) faithful adjoints. The category TopTC has inductive and projective limits; the functor U commutes with such limits.

Let I be a small category and $F:I\to$TopTC be a functor. The object (X,φ) together with the family of morphisms $f_i:F(i)\to(X,\varphi)$ $[f_i:(X,\varphi)\to F(i)]$ is the inductive [projective] limit of F iff the set X together with the family of functions $(U(f_i))$ is the inductive [projective] limit of the functor UF and the family (f_i) is terminal [initial].

The category TopTC is Cartesian closed whenever it is not an elementary topos. If $\mathscr{X}=(X,\varphi)$ and $\mathscr{Y}=(Y,\psi)$ are TC-spaces, then

$$\mathscr{Y}^{\mathscr{X}}=(\mathrm{TopTC}(\mathscr{X},\mathscr{Y}),\rho),$$

where for every $f\in\mathrm{TopTC}\ (\mathscr{X},\mathscr{Y})$, $\rho(f)=\{g\in\mathrm{TopTC}(\mathscr{X},\mathscr{Y})\mid$ for every $A\subseteq X$, $g(\varphi(A))\subseteq\varphi(f(A))\}$.

The category TopTC has general images and coimages [2], and the functor U commutes with them. In TopTC the strong, strict [4], effective and universal effective monomorphisms coincide with the initial injective morphisms.

In TopTC the strong, strict, effective and universal effective epimorphisms coincide with the terminal surjective morphisms. Also, in TopTC the isomorphisms coincide with terminal and bijective morphisms (or, equivalently, with initial and bijective morphisms).

4. Categorical properties of TK-spaces

Theorem 5. *The category of total topological spaces* (TopTK) *is a full subcategory of* TopTC; TopTK *is isomorphic with the category of preorder relations.*

We display the main properties of the category TopTK, and of the embedding functor E:TopTK$\to$TopTC.

The functor E has a faithful left adjoint, but E does not have a right adjoint.

In the category TopTK there exist inductive and projective limits. The functor E commutes only with projective limits.

In the category TopTK the monic (epic or initial) families of morphisms has similar properties as the corresponding families in TopTC.

A family of morphisms $f_i:(X_i, \varphi_i)\to(X, \mathscr{C})$ in TopTK is terminal iff the family $f_i:(X_i, \varphi_i)\to(X, \varphi)$ is terminal in TopTC and for any $A\subseteq X$, $\mathscr{C}(A)=\bigcup_{n\in N}\varphi^n(A)$.

The category TopTK is cartesian closed. TopTK has general images and coimages. The functor E commutes with general images. The functor E does not commute with general coimages, though the functor UE commutes with these objects. In TopTK the strong, strict, effective and universal effective monomorphisms coincide with the initial, injective morphisms. In TopTK the strong, strict and effective epimorphisms coincide with the terminal (in TopTK) surjective morphisms. There exist effective epimorphisms which are not universal effective.

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