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**Ludwig Staiger**
Martin-Luther-Universität
Halle-Wittenberg

**Klaus W. Wagner**
Julius-Maximilians-Universität Würzburg

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# There are Forty Nine KURATOWSKI Lattices in CANTOR Space*

***Ludwig Staiger***†
Martin-Luther-Universität Halle-Wittenberg
Institut für Informatik
von-Seckendorff-Platz 1, D–06099 Halle (Saale), Germany

and

***Klaus W. Wagner***‡
Julius-Maximilians-Universität Würzburg
Institut für Informatik
Am Hubland, D–97074 Würzburg, Germany

**Abstract**

Kuratowski observed that, starting from a subset $M$ of a topological space and applying the closure operator and the interior operator arbitrarily often, one can generate at most seven different sets. We show that there are forty nine different types of sets w.r.t. the inclusion relations between the seven generated sets. All these types really occur in Cantor space, even for subsets defined by finite automata. For a given type, it is NL-complete to decide whether a set $M$, accepted by a given finite automaton, is of this type.

In the topological space of real numbers only 39 of the 49 types really occur.

**Keywords:** topology, closure, interior, Cantor space, finite automata, NL-complete

The present paper addresses an issue relating elementary topology with automata theory. It considers, in a topological space $\mathcal{X}$, the inclusion structures, here called *types*, that can hold among the (up to) seven distinct sets a subset $M$ generates under closure $\mathrm{C}$ and interior $\mathrm{I}$. In [MMW07] it is shown

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†email: **staiger@informatik.uni-halle.de**
‡email: **klaus.wagner@uni-wuerzburg.de**

that there are 49 different such types. Using a formal derivation system the authors of [MMW07] constructed several 10-element topologies presenting all 49 types. Here we show that these 49 types can be constructed by automata theoretic means. To this end we use the regular $\omega$-languages and the setting of Cantor space.

We first derive in the general case the properties of these 49 inclusion structures, and we consider the required topological properties for sets $M$ having a certain type. In the subsequent Section 2 we show that for every of the 49 types there is a regular $\omega$-language (subset of the Cantor space) representing this type. The connection to finite automata and decision problems is the topic of the next section. Here it is shown that, for a given finite automaton, the problem whether its accepted $\omega$-language is of a certain type is NL-complete. The final section gives an example that not every space has subsets of all 49 types—the real line admits only 39 types.

# 1 Topological Spaces in General

## 1.1 Introduction

A topological space is a pair $(\mathcal{X}, \mathcal{O})$ where $\mathcal{X}$ is a non-empty set and $\mathcal{O} \subseteq 2^{\mathcal{X}}$ is a family of subsets of $\mathcal{X}$ which is closed under arbitrary union and under finite intersection. The family $\mathcal{O}$ is usually called the family of *open* subsets of the space $\mathcal{X}$. Their complements are referred to as *closed* sets of the space $\mathcal{X}$.

Kuratowski observed that topological spaces can be likewise defined using closure or interior operators. A topological interior operator $I$ is a mapping $I : 2^{\mathcal{X}} \to 2^{\mathcal{X}}$ satisfying the following relations. It assigns to a subset $M \subseteq \mathcal{X}$ the largest open set contained in $M$.

$$\begin{aligned} I\mathcal{X} &= \mathcal{X} \\ IIM &= IM \subseteq M \text{, and} \\ I(M_1 \cap M_2) &= IM_1 \cap IM_2 \end{aligned} \qquad (1)$$

Using the complementary (duality) relation between open and closed sets one defines the closure of (smallest closed set containing) $M$ as follows.

$$CM =_{\text{def}} \mathcal{X} \setminus I(\mathcal{X} \setminus M) \qquad (2)$$

Then the following holds.

$$\begin{aligned} C\emptyset &= \emptyset \\ CCM &= CM \supseteq M \\ C(M_1 \cup M_2) &= CM_1 \cup CM_2 \end{aligned} \qquad (3)$$

Since $I(M_1 \cup M_2) \cap I(\mathcal{X} \setminus M_2) = I(M_1 \setminus M_2) \subseteq IM_1$ we obtain the following (see [Kur66, RS63]).

$$I(M_1 \cup M_2) \subseteq IM_1 \cup CM_2 \subseteq CIM_1 \cup ICM_2 \tag{4}$$

In the paper [Kur22] (see also [Kur66, Ch. I, § 4]) Kuratowski proved that starting from a subset $M$ of $\mathcal{X}$ and applying $C$ and $I$ arbitrarily often, one obtains only the (not necessarily different) seven sets $M$, $CM$, $IM$, $CIM$, $ICM$, $CICM$, and $ICIM$. This can be easily verified using the following theorem.

**Theorem 1 ([Kur22])** $CICIM = CIM$ *and* $ICICM = ICM$, *for every* $M \subseteq \mathcal{X}$.

Because of the monotonicity and the idempotence of the operators $C$ and $I$ as well as the property $IM \subseteq M \subseteq CM$ we obtain the inclusion structure between these seven sets shown in Fig. 1. We will refer to this structure in the sequel as the *Kuratowski lattice* of the set $M$. More precisely, given a topological space $\mathcal{X}$ and a set $M \subseteq \mathcal{X}$ the *Kuratowski lattice* of the set $M$ is the vector $\big(M, CM, CICM, CIM, ICM, ICIM, IM\big)$, and we say that two Kuratowski lattices $(A_1, A_2, \ldots, A_7)$ and $(B_1, B_2, \ldots, B_7)$ are *isomorphic* provided $A_i \subseteq A_j \Leftrightarrow B_i \subseteq B_j$ for all $i, j \in \{1, 2, 3, 4, 5, 6, 7\}$.

The general shape of a Kuratowski lattice is depicted in Figure 1. It describes all inclusion relations between the seven sets which are necessarily fulfilled, that is, derivable from Eqs. (1) and (3) (see also [Kur66, Chapter 1, §4, V.]).

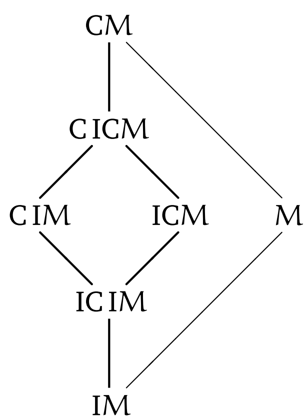


Figure 1: Kuratowski lattice

## 1.2 The Types

Depending on the particular properties of the set $M$ there might hold additional inclusions. These simplify the shape of the Kuratowski lattice. In this paper we are going to investigate how many, depending on the nature of the set $M$, non-isomorphic Kuratowski lattices might exist.

To this end we start with a list of the 14 possible additional elementary relations. The first group (A) to (E) consists of inclusion relations – upper and lower bounds – between the set $M$ and their six derived sets, and the second group (F) to (H) solely of inclusions between the derived sets.

| lower bounds | | upper bounds | |
|---|---|---|---|
| $(A_0)$ | $CM = M$ | $(A_1)$ | $IM = M$ |
| $(B_0)$ | $CICM \subseteq M$ | $(B_1)$ | $ICIM \supseteq M$ |
| $(C_0)$ | $CIM \subseteq M$ | $(C_1)$ | $ICM \supseteq M$ |
| $(D_0)$ | $ICM \subseteq M$<br>$ICM = IM$ | $(D_1)$ | $CIM \supseteq M$<br>$CIM = CM$ |
| $(E_0)$ | $ICIM \subseteq M$<br>$ICIM = IM$ | $(E_1)$ | $CICM \supseteq M$<br>$CICM = CM$ |
| relations between derived sets | | | |
| $(F_0)$ | $ICIM \supseteq CIM$<br>$ICIM = CIM$ | $(F_1)$ | $CICM \subseteq ICM$<br>$CICM = ICM$ |
| $(G)$ | $ICM \subseteq CIM$<br>$ICIM = ICM$ | | <br>$CICM = CIM$ |
| $(H)$ | $CIM \subseteq ICM$ | | |

Table 1: Possible inclusions between $M$ and its derived sets.
By Proposition 2 conditions in the same box are equivalent.

The papers [Cha62, Lev61] contain some of these equivalences and, moreover, conditions on sets $M$ to fulfil several identities like $ICM = IM$ etc.

**Proposition 2**

1. $\mathrm{ICM} \subseteq \mathrm{M} \Leftrightarrow \mathrm{ICM} = \mathrm{IM}$, *and* $\mathrm{CIM} \supseteq \mathrm{M} \Leftrightarrow \mathrm{CIM} = \mathrm{CM}$,
2. $\mathrm{ICIM} \subseteq \mathrm{M} \Leftrightarrow \mathrm{ICIM} = \mathrm{IM}$, *and* $\mathrm{CICM} \supseteq \mathrm{M} \Leftrightarrow \mathrm{CICM} = \mathrm{CM}$,
3. $\mathrm{ICIM} \supseteq \mathrm{CIM}\, \mathrm{ICIM} = \mathrm{CIM}$, *and* $\mathrm{CICM} \supseteq \mathrm{ICM} \Leftrightarrow \mathrm{CICM} = \mathrm{ICM}$,
4. $\mathrm{ICM} \subseteq \mathrm{CIM} \Leftrightarrow \mathrm{CICM} = \mathrm{CIM} \Leftrightarrow \mathrm{ICIM} = \mathrm{ICM}$

We give a short proof of the first part of Item 1, the other equivalences are proved in a similar manner.

**Proof.** If $\mathrm{ICM} \subseteq \mathrm{M}$ then $\mathrm{IICM} = \mathrm{ICM} \subseteq \mathrm{IM}$ according to Eq. (1). The other implication follows from $M \subseteq \mathrm{CM}$ also via Eq. (1) ❑

However, the 14 elementary inclusions of Table 1 are not independent. First we give a diagram of some general implications which hold true.

**Proposition 3** *Let* $\alpha \in \{0, 1\}$. *Then the following general implication structure holds true.*

$$\begin{array}{ccccccc} & & & & & \nearrow & \mathrm{G} \\ & & & \nearrow & \mathrm{D}_\alpha & & \\ \mathrm{A}_\alpha & \rightarrow & \mathrm{B}_\alpha & & & \searrow & \mathrm{E}_\alpha \\ & & & \searrow & \mathrm{C}_\alpha & \nearrow & \end{array} \tag{5}$$

$$\mathrm{F}_\alpha \rightarrow \mathrm{H} \tag{6}$$

Next, we present some further implications which are needed in the sequel.

**Proposition 4** *Let* $\alpha \in \{0, 1\}$.

$$\begin{array}{rcl} \mathrm{D}_\alpha & \longleftrightarrow & \mathrm{E}_\alpha \wedge \mathrm{G} \quad (7) \\ \mathrm{B}_\alpha & \longleftrightarrow & \mathrm{C}_\alpha \wedge \mathrm{G} \quad (8) \\ \mathrm{G} & \longrightarrow & (\mathrm{B}_\alpha \leftrightarrow \mathrm{C}_\alpha) \wedge (\mathrm{D}_\alpha \leftrightarrow \mathrm{E}_\alpha) \quad (9) \\ \mathrm{F}_\alpha & \longrightarrow & (\mathrm{C}_\alpha \leftrightarrow \mathrm{E}_\alpha) \quad (10) \\ \mathrm{C}_0 \wedge \mathrm{C}_1 & \longrightarrow & \mathrm{H} \quad (11) \\ \mathrm{C}_\alpha \wedge \mathrm{D}_{1-\alpha} & \longrightarrow & \mathrm{A}_\alpha \quad (12) \\ \mathrm{G} \wedge \mathrm{H} & \longrightarrow & \mathrm{F}_0 \wedge \mathrm{F}_1 \quad (13) \end{array}$$

**Proof.** For Eqs. (7) and (8) the direction from left to right is in Eq. (5). The other directions and Eq. (9) follow from the identities in Item $(\mathrm{G})$ of Table 1.

In a similar way the identities in the Items $(F_\alpha)$ imply the equivalences of $(C_\alpha)$ and $(E_\alpha)$.

To prove Eq. (12), for $\alpha = 0$ we have $CIM \subseteq M$ and $M \subseteq CIM$. Thus $M = CIM$ which implies that $M$ is closed. The case $\alpha = 1$ is similar.

Eq. (13) is obvious. ❏

All in all, there are $2^{14}$ possible combinations of the 14 conditions. In the rest of this section we show that, using the implications from Proposition 3 and 4, only 49 combinations can satisfy these conditions. Thus, we obtain at most 49 different Kuratowski lattices.

We split our proof into four groups according to whether the conditions $G$ and $H$ hold or do not hold. In what follows, for $\Gamma \in \{A_0.A_1, B_0, \ldots, G, H\}$ we write $\Gamma = 1(0)$ if $\Gamma$ holds (does not hold, respectively) for the set $M$ under consideration.

### 1.2.1 The Case $\neg G \wedge \neg H$

This is the only case where $CIM$ and $ICM$ are incomparable.

According to Eqs. (5) and (6) we have $A_\alpha = B_\alpha = D_\alpha = F_\alpha = 0$ and $C_\alpha \to E_\alpha$ for $\alpha \in \{0, 1\}$, and $(C_0 = 0 \vee C_1 = 0)$ from Eq. (11) . This yields the following eight combinations listed in Table 2.[1]

| type of M | $A_0$ | $A_1$ | $B_0$ | $B_1$ | $C_0$ | $C_1$ | $D_0$ | $D_1$ | $E_0$ | $E_1$ | $F_0$ | $F_1$ | $G$ | $H$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2a | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | 0 | 0 | 0 | 0 | 0 |
| 2b | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | 0 | 0 | 0 | 0 |
| 3a | 0 | 0 | 0 | 0 | **1** | 0 | 0 | 0 | **1** | 0 | 0 | 0 | 0 | 0 |
| 3b | 0 | 0 | 0 | 0 | 0 | **1** | 0 | 0 | 0 | **1** | 0 | 0 | 0 | 0 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **1** | 0 | 0 | 0 | 0 |
| 5a | 0 | 0 | 0 | 0 | **1** | 0 | 0 | 0 | **1** | **1** | 0 | 0 | 0 | 0 |
| 5b | 0 | 0 | 0 | 0 | 0 | **1** | 0 | 0 | **1** | **1** | 0 | 0 | 0 | 0 |

Table 2: The case $\neg G \wedge \neg H$

### 1.2.2 The Case $\neg G \wedge H$

Here we have $CIM \subset ICM$.

As in the previous case, Eq. (5) implies $A_\alpha = B_\alpha = D_\alpha = 0$ and $C_\alpha \to E_\alpha$ for $\alpha \in \{0, 1\}$. If $F_\alpha = 1$ we have additionally $C_\alpha = E_\alpha$ by Eq. (10).

[1] For better orientation the ones in this and the following three tables are set in boldface.

This gives nine combinations in the case $(F_0, F_1) = (0, 0)$, six combinations in each of the cases $(F_0, F_1) \in \{(0, 1), (1, 0)\}$, and four combinations in the case $(F_0, F_1) = (1, 1)$ as shown in Table 3.

| type | M fulfils | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| of M | $A_0$ | $A_1$ | $B_0$ | $B_1$ | $C_0$ | $C_1$ | $D_0$ | $D_1$ | $E_0$ | $E_1$ | $F_0$ | $F_1$ | G | H |
| 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** |
| 7a | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | 0 | 0 | 0 | 0 | **1** |
| 7b | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | 0 | 0 | 0 | **1** |
| 8a | 0 | 0 | 0 | 0 | **1** | 0 | 0 | 0 | **1** | 0 | 0 | 0 | 0 | **1** |
| 8b | 0 | 0 | 0 | 0 | 0 | **1** | 0 | 0 | 0 | **1** | 0 | 0 | 0 | **1** |
| 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **1** | 0 | 0 | 0 | **1** |
| 10a | 0 | 0 | 0 | 0 | **1** | 0 | 0 | 0 | **1** | **1** | 0 | 0 | 0 | **1** |
| 10b | 0 | 0 | 0 | 0 | 0 | **1** | 0 | 0 | **1** | **1** | 0 | 0 | 0 | **1** |
| 11 | 0 | 0 | 0 | 0 | **1** | **1** | 0 | 0 | **1** | **1** | 0 | 0 | 0 | **1** |
| 12a | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | 0 | **1** |
| 13a | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | 0 | 0 | **1** | 0 | **1** |
| 14a | 0 | 0 | 0 | 0 | **1** | 0 | 0 | 0 | **1** | 0 | 0 | **1** | 0 | **1** |
| 15a | 0 | 0 | 0 | 0 | 0 | **1** | 0 | 0 | 0 | **1** | 0 | **1** | 0 | **1** |
| 16a | 0 | 0 | 0 | 0 | 0 | **1** | 0 | 0 | **1** | **1** | 0 | **1** | 0 | **1** |
| 17a | 0 | 0 | 0 | 0 | **1** | **1** | 0 | 0 | **1** | **1** | 0 | **1** | 0 | **1** |
| 12b | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | 0 | 0 | **1** |
| 13b | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **1** | 0 | 0 | **1** |
| 14b | 0 | 0 | 0 | 0 | 0 | **1** | 0 | 0 | 0 | **1** | **1** | 0 | 0 | **1** |
| 15b | 0 | 0 | 0 | 0 | **1** | 0 | 0 | 0 | **1** | 0 | **1** | 0 | 0 | **1** |
| 16b | 0 | 0 | 0 | 0 | **1** | 0 | 0 | 0 | **1** | **1** | **1** | 0 | 0 | **1** |
| 17b | 0 | 0 | 0 | 0 | **1** | **1** | 0 | 0 | **1** | **1** | **1** | 0 | 0 | **1** |
| 18 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **1** | 0 | **1** |
| 19a | 0 | 0 | 0 | 0 | **1** | 0 | 0 | 0 | **1** | 0 | **1** | **1** | 0 | **1** |
| 19b | 0 | 0 | 0 | 0 | 0 | **1** | 0 | 0 | 0 | **1** | **1** | **1** | 0 | **1** |
| 20 | 0 | 0 | 0 | 0 | **1** | **1** | 0 | 0 | **1** | **1** | **1** | **1** | 0 | **1** |

Table 3: The case $\neg G \wedge H$

### 1.2.3 The Case $G \wedge \neg H$

Here we have $ICM \subset CIM$.

In view of Eqs. (5), (6), (11), (9), and (12) we get $A_\alpha \to B_\alpha \to D_\alpha$, $F_0 = F_1 = 0$, $(B_0 = 0 \vee B_1 = 0)$, $B_\alpha = C_\alpha$, $D_\alpha = E_\alpha$, and $C_\alpha \wedge D_{1-\alpha} \to A_\alpha$. This results in the ten possible combinations shown in Table 4.

| type of M | $A_0$ | $A_1$ | $B_0$ | $B_1$ | $C_0$ | $C_1$ | $D_0$ | $D_1$ | $E_0$ | $E_1$ | $F_0$ | $F_1$ | $G$ | $H$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 21 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | 0 |
| 22a | 0 | 0 | 0 | 0 | 0 | 0 | **1** | 0 | **1** | 0 | 0 | 0 | **1** | 0 |
| 22b | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | 0 | **1** | 0 | 0 | **1** | 0 |
| 23a | 0 | 0 | **1** | 0 | **1** | 0 | **1** | 0 | **1** | 0 | 0 | 0 | **1** | 0 |
| 23b | 0 | 0 | 0 | **1** | 0 | **1** | 0 | **1** | 0 | **1** | 0 | 0 | **1** | 0 |
| 24a | **1** | 0 | **1** | 0 | **1** | 0 | **1** | 0 | **1** | 0 | 0 | 0 | **1** | 0 |
| 24b | 0 | **1** | 0 | **1** | 0 | **1** | 0 | **1** | 0 | **1** | 0 | 0 | **1** | 0 |
| 25 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **1** | **1** | **1** | 0 | 0 | **1** | 0 |
| 26a | **1** | 0 | **1** | 0 | **1** | 0 | **1** | **1** | **1** | **1** | 0 | 0 | **1** | 0 |
| 26b | 0 | **1** | 0 | **1** | 0 | **1** | **1** | **1** | **1** | **1** | 0 | 0 | **1** | 0 |

Table 4: The case $G \wedge \neg H$

### 1.2.4 The Case $G \wedge H$

Here we have $CIM = ICM$.

Because of Eq. (13) we have $F_0 = F_1 = 1$. Now, from Eqs. (10) and (9) we conclude $B_\alpha = C_\alpha = D_\alpha = E_\alpha$. By Eqs. (5) and (12) we get $A_\alpha \rightarrow B_\alpha$ and $(B_0 \wedge B_1) \rightarrow (A_0 \wedge A_1)$. Table 5 shows the resulting six possible combinations.

| type of M | $A_0$ | $A_1$ | $B_0$ | $B_1$ | $C_0$ | $C_1$ | $D_0$ | $D_1$ | $E_0$ | $E_1$ | $F_0$ | $F_1$ | $G$ | $H$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 27 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | **1** | **1** | **1** | **1** |
| 28a | 0 | 0 | **1** | 0 | **1** | 0 | **1** | 0 | **1** | 0 | **1** | **1** | **1** | **1** |
| 28b | 0 | 0 | 0 | **1** | 0 | **1** | 0 | **1** | 0 | **1** | **1** | **1** | **1** | **1** |
| 29a | **1** | 0 | **1** | 0 | **1** | 0 | **1** | 0 | **1** | 0 | **1** | **1** | **1** | **1** |
| 29b | 0 | **1** | 0 | **1** | 0 | **1** | 0 | **1** | 0 | **1** | **1** | **1** | **1** | **1** |
| 30 | **1** | **1** | **1** | **1** | **1** | **1** | **1** | **1** | **1** | **1** | **1** | **1** | **1** | **1** |

Table 5: The case $G \wedge H$

So far it is not yet clear that each of the 49 types can really occur in a topological space. In the following we will see that they can occur in the Cantor space. Before we proceed to this goal we discuss the topological complexity which is necessary for a subset $M \subseteq \mathcal{X}$ to generate a Kuratowski lattice of a certain type.

## 1.3 Duality

First we refer again to the duality of the operators C and I. From Eq. (2) we known that for the operations C and I the duality principle $\mathcal{X} \setminus CM = I(\mathcal{X} \setminus M)$ holds. This duality principle carries over to the conditions of Table 1.

**Proposition 5**

**Duality** *Let* $\Gamma \in \{A, B, C, D, E, F\}$.
*Condition* $\Gamma_0$ *holds for* M *if and only if condition* $\Gamma_1$ *holds for* $\mathcal{X} \setminus M$.

**Self-Duality** *Let* $\Gamma \in \{G, H\}$.
*Condition* $\Gamma$ *holds for* M *if and only if condition* $\Gamma$ *holds for* $\mathcal{X} \setminus M$.

**Proof.** The first part follows from the duality relation $C(\mathcal{X} \setminus M) = \mathcal{X} \setminus IM$ (see Eq. (2)).

For the second, applying the duality relation Eq. (2) twice we obtain $IC(\mathcal{X} \setminus M) = I(\mathcal{X} \setminus IM) = \mathcal{X} \setminus CIM$ and $CI(\mathcal{X} \setminus M) = C(\mathcal{X} \setminus CM) = \mathcal{X} \setminus ICM$. Then, in case $\Gamma = G$, the hypothesis $ICM \subseteq CIM$ ($CIM \subseteq ICM$, respectively) yields the assertion. ❏

Due to the duality between conditions shown in Proposition 5 there is a duality between types in the Tables 2, 3, 4 and 5.

**Proposition 6** 1. *Let* $\tau \in \{1, 4, 6, 9, 11, 18, 20, 21, 25, 27, 30\}$.
*The set* M *is of type* $\tau$ *if and only* $\mathcal{X} \setminus M$ *is of type* $\tau$.

2. *Let* $\tau \in \{1, 2, \ldots, 30\} \setminus \{1, 4, 6, 9, 11, 18, 20, 21, 25, 27, 30\}$.
*The set* M *is of type* $\tau a$ *if and only* $\mathcal{X} \setminus M$ *is of type* $\tau b$.

## 1.4 Topological structure

In this section we investigate the necessary topological structure for sets $M \subseteq \mathcal{X}$ to be of a certain type $\tau$. Let for a topological space $\mathcal{X}$ be $\mathbf{F} =_{def} \{M \mid M \subseteq \mathcal{X} \wedge M \text{ is closed}\}$ and $\mathbf{G} = \{M \mid M \subseteq \mathcal{X} \wedge M \text{ is open}\}$ the families of closed an open subsets, respectively. Moreover, define $\mathbf{F} \vee \mathbf{G} =_{def} \{F \cup E \mid F \in \mathbf{F} \wedge E \in \mathbf{G}\}$ and $\mathbf{F} \wedge \mathbf{G} =_{def} \{F \cap E \mid F \in \mathbf{F} \wedge E \in \mathbf{G}\}$, and let, as usual, $\mathbf{F}_\sigma$ be the set of countable unions of closed sets.

First we consider the topologically simple types.

**Lemma 7** 1. M *is of type 30 if and only if* M *is open and closed.*

2. M *is of type 24a, 26a or 29a if and only if* M *is closed, but not open.*

3. M *is of type 24b, 26b or 29b if and only if* M *is open, but not closed.*

**Proof.** Since $M$ is closed if and only if $M$ fulfils $(A_0)$ and $M$ is open if and only if $M$ fulfil-ls $(A_1)$, the proof follows from inspecting the Tables 2 to 5. ❏

Theorem 7.2 and 7.3 can be made more precise.

**Corollary 8** *1. $M$ is of type 26a if and only if $M = \mathcal{C}\mathcal{I}M$ and $M$ is not open.*[2]

*2. $M$ is of type 29a if and only if $\mathcal{I}M = \mathcal{C}\mathcal{I}M$ and $M$ is closed but not open.*

*3. $M$ is of type 26b if and only if $M = \mathcal{I}\mathcal{C}M$ and $M$ is not closed.*[3]

*4. $M$ is of type 29b if and only if $\mathcal{C}M = \mathcal{I}\mathcal{C}M$ and $M$ is open but not closed.*

**Proof.** 1. A look at Tables 2, 3, 4, and 5 shows that $M$ is of type 26a if and only if it satisfies $(C_0)$ and $(D_1)$ but not $(A_1)$. But $(C_0)$ and $(D_1)$ is equivalent to $M = \mathcal{C}\mathcal{I}M$, and $(A_1)$ is equivalent to $M$ being open.

2. A look at Tables 2, 3, 4, and 5 shows that $M$ is of type 29a if and only if it satisfies $(A_0)$,$(E_0)$, and $(F_0)$ but not $(A_1)$. But $(E_0)$ and $(F_0)$ is equivalent to $\mathcal{I}M = \mathcal{C}\mathcal{I}M$, $(A_0)$ is equivalent to $M$ being closed, and $(A_1)$ is equivalent to $M$ being open.

The assertions 3 and 4 follow by duality. ❏

For the structure of the sets of the remaining 42 types the following notion is helpful. We call a set $M \subseteq \mathcal{X}$ *nowhere dense* provided $\mathcal{I}\mathcal{C}M = \emptyset$, that is, if the closure $\mathcal{C}M$ does not contain a non-empty open set. Clearly, this condition is equivalent to $\mathcal{C}\mathcal{I}\mathcal{C}M = \emptyset$.

As an immediate consequence we obtain the following relation to the types 28a, 29a and 30.

**Proposition 9** *Let $M \subseteq \mathcal{X}$ be nowhere dense. Then*

*1. $M$ is of type 28a, 29a, or 30.*

*2. $M$ is of type 30 if and only if $M = \emptyset$.*

*3. $M$ is of type 29a if and only if $M \neq \emptyset$ and $M$ is closed, and*

*4. $M$ is of type 28a if and only if $M$ is not closed.*

The papers [Cha62, Lev61] show that equality up to nowhere dense sets (cf. Theorem 10.2) is related to several of the identities in Table 1.

Moreover, it holds the following (cf. with [Cha62, Theorem 4] and [Kur66, Chapter 1.V]).

[2] Sets satisfying $M = \mathcal{C}\mathcal{I}M$ are also known as *closed domains*.
[3] Sets satisfying $M = \mathcal{I}\mathcal{C}M$ are also known as *open domains*.

**Theorem 10** *Let $\mathcal{X}$ be a topological space.*

1. *The family $\mathcal{G} =_{\mathrm{def}} \{M \mid M \subseteq \mathcal{X} \wedge ICM \subseteq CIM\}$ is a Boolean algebra which contains all open (and closed) and all nowhere dense subsets of $\mathcal{X}$.*

2. *$M \in \mathcal{G}$ if and only if there is an open set $P \in \mathcal{X}$ such that $M \setminus P$ and $P \setminus M$ are nowhere dense.*

For the sake of completeness we give a proof.
**Proof.** (1) Obviously, the family $\mathcal{G}$ contains all open and all nowhere dense subsets of $\mathcal{X}$. By Proposition 5.2 the family $\mathcal{G}$ is closed under complementation.

In order to show closure under union we observe that due to the monotonicity and idempotence of the operators C and I and Eq. (4) $IC(M_1 \cup M_2) \subseteq ICM_1 \cup CICM_2$.

Then Proposition 2 (G) and the hypothesis $ICM_1 \subseteq CIM_1$ yield the assertion $IC(M_1 \cup M_2) \subseteq CIM_1 \cup CIM_2 \subseteq CI(M_1 \cup M_2)$.

(2) Assume, $M \setminus P$ and $P \setminus M$ be nowhere dense for some open set $P$. Then, in view of $IC(M \setminus P) = \emptyset$, Eq. (4) shows $ICM \subseteq I(CP \cup C(M \setminus P)) \subseteq CICP \cup IC(M \setminus P) = CICP$.

As $P$ is open and $P \setminus M$ is nowhere dense, we have, again using Eq. (4), $CICP \subseteq CICI(M \cup (P \setminus M)) \subseteq CIC(CIM \cup IC(P \setminus M)) = CIM$.

Conversely, let $M \in \mathcal{G}$. Since $IM \in \mathcal{G}$, we have $M \setminus IM \in \mathcal{G}$ and, consequently, $IC(M \setminus IM) \subseteq CI(M \setminus IM) = \emptyset$, as $I(M \setminus IM) = \emptyset$. Thus $M \setminus IM$ and $IM \setminus M = \emptyset$ are nowhere dense. ❏

The last part of the preceding proof shows the following.

**Corollary 11** *$M \in \mathcal{G}$ if and only if $M \setminus IM$ is nowhere dense, and if $M \in \mathcal{G}$ then $M$ contains a non-empty open subset or $M$ is nowhere dense.*

Now we show that unions of open and closed sets cannot be of type 23a and 28a unless they are closed. By the duality principle this is equivalent to the fact the intersections of open and closed sets cannot be of type 23b and 28b unless they are open.

**Theorem 12** *Let $\mathcal{X}$ be a topological space, $M_1 \subseteq \mathcal{X}$ open and $M_2 \subseteq \mathcal{X}$ closed.*

1. *If $M = M_1 \cup M_2$ and $CIM \subseteq M$ then $M$ is closed.*

2. *If $M = M_1 \cap M_2$ and $ICM \supseteq M$ then $M$ is open.*

**Proof.** We prove only the first assertion, the second follows by the duality principle.

Consider $CM = C(M_1 \cup M_2) = CM_1 \cup CM_2$. Then $CM_2 = M_2$ and, since $M_1$ is open, we have $M_1 \subseteq IM$. So $CM \subseteq CIM \cup M_2 = M$ is closed. ❏

None of the sets $M$ of types $1, \ldots, 20$ satisfies $ICM \subseteq CIM$. Thus from Lemma 7 and Theorems 10, and 12 we obtain the following corollary.

**Corollary 13** *1. A set of type* $1, \ldots, 20$ *cannot be in the class* $\mathcal{G}$.

*2. A set of type 21, 22a, 22b, 25 or 27 cannot be in* $\mathbf{F} \cup \mathbf{G}$.

*3. A set of type 23a or 28a cannot be in* $\mathbf{F} \vee \mathbf{G}$.

*4. A set of type 23b or 28b cannot be in* $\mathbf{F} \wedge \mathbf{G}$

In Section 2.3 we will see that these lower bounds cannot be improved.

# 2 The Cantor Space

## 2.1 Languages of infinite words

The Cantor space may be introduced conveniently using the notation known from Formal Language Theory. Let $X$ be an alphabet of cardinality $|X| = r \geqslant 2$. Then $X^*$ is the set of finite words on $X$, including the *empty word* $e$, and $X^\omega$ is the set of infinite strings ($\omega$-words) over $X$. Subsets of $X^*$ will be referred to as *languages* and subsets of $X^\omega$ as $\omega$*-languages*.

For $w \in X^*$ and $\eta \in X^* \cup X^\omega$ let $w \cdot \eta$ be their *concatenation*. This concatenation product extends in an obvious way to subsets $W \subseteq X^*$ and $M \subseteq X^* \cup X^\omega$. For a language $W$ let $W^* =_{\mathrm{def}} \bigcup_{i=0}^{\infty} W^i$, and $W^\omega =_{\mathrm{def}} \{w_1 \cdots w_i \cdots \mid w_i \in W \setminus \{e\}\}$ be the set of infinite strings formed by concatenating non-empty words in $W$. If $W = \{w\}, w \neq e$, we will sometimes write $w^*$ and $w^\omega$ instead of $\{w\}^*$ and $\{w\}^\omega$, respectively. Furthermore, $\mathbf{pref}(M)$ is the set of all finite prefixes of strings in $M \subseteq X^* \cup X^\omega$. We shall abbreviate $w \in \mathbf{pref}(\{\eta\})$ $(\eta \in X^* \cup X^\omega)$ by $w \sqsubseteq \eta$.

As usual, we consider $X^\omega$ as a topological space (Cantor space). The closure of a subset $M \subseteq X^\omega, CM$, is described as $CM =_{\mathrm{def}} \{\xi \mid \mathbf{pref}(\{\xi\}) \subseteq \mathbf{pref}(M)\}$. The *open sets* in Cantor space are the $\omega$-languages of the form $W \cdot X^\omega$. Accordingly, $IM = \bigcup\{w \cdot X^\omega \mid w \cdot X^\omega \subseteq M\}$ is the interior of $M \subseteq X^\omega$.

For the purposes of our paper it is convenient to represent certain subsets of the Cantor Space as regular $\omega$-languages, that is, $\omega$-languages defined by finite automata. To this end we mention that a language $W \subseteq X^*$ is *regular* if it can be obtained from finite subsets of $X^*$ by a finite number of applications of the operations $\cup, \cdot$, and $^*$; and s subset $M \subseteq X^\omega$ is a *regular $\omega$-language* if it is of the form $M = \bigcup_{i=1}^n W_i \cdot V_i^\omega$ where $W_i, V_i \subseteq X^*$ are regular languages. The relation between regular $\omega$-languages and finite automata will be explains in Section 2.4.

We assume the reader to be familiar with the basic facts of the theory of regular languages and finite automata. For more details on $\omega$-languages and regular $\omega$-languages see the book [PP04] or the survey papers [Sta97, Tho90].

## 2.2 All Types Exist in Cantor Space

In this section we will show that in Cantor space there are really 49 different types of sets.

The following proposition is very helpful because it enables us to construct (sets of) new types from other (given) types.

For a set $M \subseteq X^\omega$ and a $\Gamma \in \{A_0, A_1, B_0, \ldots, G, H\}$ we write $M(\Gamma) = 1(0)$ if $M$ fulfils $\Gamma$ (does not fulfil $\Gamma$, respectively). Furthermore, we say that $M$ is of type $\tau = (M(A_0), M(A_1), M(B_0), \ldots, , M(G), M(H))$.

**Proposition 14** *Let $M_0, M_1 \subseteq X^\omega$ and $a, b \in X$ such that $a \neq b$.*

1. *If $M_0$ is of type $(\alpha_1, \alpha_2, \ldots, \alpha_{14})$ and $M_1$ is of type $(\beta_1, \beta_2, \ldots, \beta_{14})$ then $aM_0 \cup bM_1$ is of type $(\alpha_1 \wedge \beta_1, \alpha_2 \wedge \beta_2, \ldots, \alpha_{14} \wedge \beta_{14})$.*

2. *Moreover, if $M_0, M_1$ are both in one of the classes* $\mathbf{F}, \mathbf{G}, \mathbf{F}\vee\mathbf{G}, \mathbf{F}\wedge\mathbf{G}, \mathbf{F}_\sigma$ *or $\mathcal{G}$ then $aM_0 \cup bM_1$ belongs also to the same class.*

3. *If $M_0 \notin \mathcal{G}$ then $aM_0 \cup bM_1 \notin \mathcal{G}$.*

**Proof.** The first assertion is an immediate consequence of $C(aM_0 \cup bM_1) = aCM_0 \cup bCM_1$, $I(aM_0 \cup bM_1) = a\,IM_0 \cup b\,IM_1$, and $aM_0 \cup bM_1 \subseteq aP_0 \cup bP_1$ if and only if $M_0 \subseteq P_0$ and $M_1 \subseteq P_1$.

The second assertion is obvious for classes closed under union. So it suffices to prove it for the class $\mathbf{F}\wedge\mathbf{G}$. Let $M_i = Q_i \cap P_i$, $i = 0, 1$, where $Q_0, Q_1$ are closed and $P_0, P_1$ are open. Now the assertion follows from the identity $aM_0 \cup bM_1 = a \cdot (Q_0 \cap P_0) \cup b \cdot (Q_1 \cap P_1) = (a \cdot Q_0 \cup b \cdot Q_1) \cap (a \cdot P_0 \cup b \cdot P_1)$.

The third one follows from $aM_0 = (aM_0 \cup bM_1) \cap a \cdot X^\omega$, and $aM \in \mathcal{G}$ if and only if $M \in \mathcal{G}$. ❏

For types $(\alpha_1, \alpha_2, \ldots, \alpha_{14})$ and $(\beta_1, \beta_2, \ldots, \beta_{14})$ let
$(\alpha_1, \alpha_2, \ldots, \alpha_{14}) \wedge (\beta_1, \beta_2, \ldots, \beta_{14}) =_{\text{def}} (\alpha_1 \wedge \beta_1, \alpha_2 \wedge \beta_2, \ldots, \alpha_{14} \wedge \beta_{14})$.
We observe:

**Proposition 15** *Let* $x \in \{a, b\}$*, and put* $\overline{a} =_{\text{def}} b$ *and* $\overline{b} =_{\text{def}} a$*. Then*

*1.* $(1) = (6) \wedge (21)$
*2.* $(2x) = (3x) \wedge (4)$
*3.* $(3x) = (5x) \wedge (8x)$
*4.* $(4) = (5a) \wedge (5b)$
*5.* $(5x) = (17b) \wedge (26x)$
*6.* $(6) = (11) \wedge (27)$
*7.* $(7x) = (8x) \wedge (9)$
*8.* $(8x) = (11) \wedge (29x)$
*9.* $(9) = (10a) \wedge (10b)$
*10.* $(10x) = (11) \wedge (16\overline{x})$
*11.* $(11) = (17a) \wedge (17b)$
*12.* $(12x) = (17x) \wedge (27)$
*13.* $(13x) = (14x) \wedge (16x)$
*14.* $(14x) = (17x) \wedge (29x)$
*15.* $(15x) = (17x) \wedge (29\overline{x})$
*18.* $(18) = (20) \wedge (27)$
*19.* $(19x) = (20) \wedge (29x)$
*21.* $(21) = (25) \wedge (27)$
*22.* $(22x) = (25) \wedge (29x)$
*23.* $(23x) = (26x) \wedge (28x)$
*24.* $(24x) = (26x) \wedge (29x)$
*25.* $(25) = (26a) \wedge (26b)$
*27.* $(27) = (29a) \wedge (29b)$

The types $(16a)$, $(16b)$, $(17a)$, $(17b)$, $(20)$, $(26a)$, $(26b)$, $(28a)$, $(28b)$, $(29a)$, $(29b)$, and $(30)$ are missing on the left hand sides of the equations in Proposition 15. We will refer to them as *basic types*.

Every other type is the $\wedge$-combination of basic types or types having a higher number. So, if we can show that the basic types exist in Cantor space then all 49 types exist in Cantor space. Because of Proposition 6 it is sufficient to prove that the types $(16a)$, $(17a)$, $(20)$, $(26a)$, $(28a)$, $(29a)$, and $(30)$ do exist in Cantor space.

*Remark.* In most of the cases in Proposition 15 other combinations of compound types are possible. We have chosen the present ones for reasons which will become apparent in Sections 2.3 and 3.2.

**Lemma 16** *Let* $X = \{0, 1\}$*.*

1. *The set* $M_{16} =_{\text{def}} 0^*11\{0,1\}^\omega \cup 0^*10\{0,1\}^*0^\omega$ *is of type 16a.*
2. *The set* $M_{17} =_{\text{def}} 0^\omega \cup 0^*11\{0,1\}^\omega \cup 0^*10\{0,1\}^*0^\omega$ *is of type 17a.*
3. *The set* $M_{20} =_{\text{def}} \{0,1\}^*0^\omega$ *is of type 20.*
4. *The set* $M_{26} =_{\text{def}} 0^\omega \cup 0^*11\{0,1\}^\omega$ *is of type 26a.*
5. *The set* $M_{28} =_{\text{def}} 0^*10^\omega$ *is of type 28a.*
6. *The set* $M_{29} =_{\text{def}} 0^\omega$ *is of type 29a.*
7. *The set* $M_{30} =_{\text{def}} \emptyset$ *is of type 30.*

**Proof.** For $M_{16}, M_{17}$ and $M_{20}$ we have $\mathbf{pref}(M_i) = \{0,1\}^*$. Consequently, $CM_i = ICM_i = CICM_i = \{0,1\}^\omega$ for $i = 16, 17$ or $20$.

1. Here $IM_{16} = 0^*11\{0,1\}^\omega$ whence $CIM_{16} = 0^\omega \cup 0^*11\{0,1\}^\omega$ and $ICIM_{16} = IM_{16}$. It is now obvious that $IM_{16} \subset CIM_{16} \subset ICM_{16}$, $IM_{16} \subset M_{16} \subset ICM_{16}$ and neither $M_{16} \subseteq CIM_{16}$ nor $CIM_{16} \subseteq M_{16}$.

   Thus conditions $(C_1), (E_0), (E_1), (F_1)$ and $(H)$ hold true whereas $(C_0)$ and $(G)$ are false. The rest follows from Eqs. (5) and (10).

2. Here we have also $\mathrm{IM}_{17} = \mathrm{ICIM}_{17} = 0^*11\{0,1\}^\omega$. The rest follows from $\mathrm{ICM}_{17} \supset M_{17} \supset \mathrm{CIM}_{17} = 0^\omega \cup 0^*11\{0,1\}^\omega \supset \mathrm{IM}_{17}$ as in Item 1.

3. We have $\mathrm{IM}_{20} = \emptyset$. Then $\mathrm{IM}_{20} = \mathrm{CIM}_{20} = \mathrm{ICIM}_{20} = \emptyset \subset M_{20} \subset \mathrm{ICM}_{20}$.

   Thus conditions $(C_\alpha), (E_\alpha), (F_\alpha), \alpha \in \{0,1\}$, and $(H)$ hold true whereas $(G)$ is false. The rest follows from Eq. (5).

4. $M_{26} = \mathrm{CIM}_{16}$ whence $M_{26} = \mathrm{CIM}_{26}$. Since $M_{26}$ is closed but not open, the assertion follows with Corollary 8.

The remaining three sets $M_{28}, M_{29}$ and $M_{30}$ are nowhere dense, so the assertion follows with Proposition 9. ❏

*Remark.* Analogous considerations show that the countable $\omega$-languages $M'_{16} =_{\mathrm{def}} 0^\omega \cup 0^*10\{0,1\}^*0^\omega$ and $M'_{17} =_{\mathrm{def}} 0^*10\{0,1\}^*0^\omega$ are of types $(16b)$ or $(17b)$, respectively.

As a consequence of Propositions 14, 15 and Lemma 16 we obtain

**Theorem 17** *All forty-nine types do exist in Cantor space.*

### 2.3 Topological complexity

Here we show that, in the Cantor space, the results in Corollary 13 are optimal.

**Lemma 18** *1.* $M_{30} \in \mathbf{F} \cap \mathbf{G}$

*2.* $M_{26}, M_{29} \in \mathbf{F}$

*3.* $M_{28} \in (\mathbf{F} \wedge \mathbf{G})$

*4.* $M_{16}, M_{17}, M_{20} \in \mathbf{F}_\sigma$

**Proof.** The first two items are obvious. $M_{28} = (0^\omega \cup 0^*10^\omega) \cap 0^*1\{0,1\}^\omega$ shows that $M_{28}$ is the intersection of a closed with an open set.. The last assertion follows from $0^*11\{0,1\}^\omega = \bigcup_{i=0}^{\infty} 0^i 11\{0,1\}^\omega$ and the fact that $0^*10\{0,1\}^*0^\omega$ is a countable set. ❏

Combining with the results of the preceding section we obtain the following.

**Theorem 19** *Let* $\mathcal{M}_\tau$ *be the family* $\{M \mid M \subseteq X^\omega \wedge M$ *is of type* $\tau\}$.

1. *For each* $\tau \in \{1,\ldots,20\}$, *there exists a regular* $\omega$*-language* $M \in \mathbf{F}_\sigma \cap \mathcal{M}_\tau$, *but* $\mathcal{M}_\tau \cap \mathcal{G} = \emptyset$.

2. *For each* $\tau \in \{21, 22a, 22b, 25, 27\}$*, there exists a regular* $\omega$*-language* $M \in (\mathbf{F}\wedge\mathbf{G}) \cap (\mathbf{F}\vee\mathbf{G})$ *of type* $\tau$*, but there does not exist an open set or a closed set of type* $\tau$*.*

3. *For* $\tau \in \{23a, 28a\}$*, there exists a regular* $\omega$*-language in* $(\mathbf{F}\wedge\mathbf{G})$ *of type* $\tau$*, but* $\mathcal{M}_\tau \cap (\mathbf{F}\vee\mathbf{G}) = \emptyset$*.*

4. *For* $\tau \in \{23b, 28b\}$*, there exists a regular* $\omega$*-language in* $(\mathbf{F}\vee\mathbf{G})$ *of type* $\tau$*, but* $\mathcal{M}_\tau \cap (\mathbf{F}\wedge\mathbf{G}) = \emptyset$*.*

**Proof.** The lower bounds follow from Corollary 13. It remains to show that there are regular $\omega$-language in the respective classes.

All basic types contain regular $\omega$-languages in $\mathbf{F}_\sigma$. Using Proposition 14.2 and Proposition 15 one can successively show that all types contain regular $\omega$-languages in $\mathbf{F}_\sigma$.

The sets $M_{26}$ and $M_{29}$ are in $(\mathbf{F}\wedge\mathbf{G}) \cap (\mathbf{F}\vee\mathbf{G})$, in fact, they are closed. Thus Proposition 5, Proposition 14.2 and Proposition 15.27, 15.25, 15.22 and 15.21 show that $\mathcal{M}_\tau$, $\tau \in \{27, 25, 22a, 22b, 21\}$, contain sets in $(\mathbf{F}\wedge\mathbf{G}) \cap (\mathbf{F}\vee\mathbf{G})$.

The proof for $\mathcal{M}_{23a}$ and $\mathcal{M}_{28a}$ is obtained similarly utilising the fact that $M_{26}, M_{28} \in \mathbf{F}\wedge\mathbf{G}$. The remaining assertion is dual to the previous one. ❏

### 2.4 Regular $\omega$-languages and finite automata

An $\omega$-language $M \subseteq X^\omega$ is regular provided there are a finite (deterministic) automaton $\mathcal{A} = (X; S; s_0; \delta)$ and a table $\mathcal{T} \subseteq \{S' \mid S' \subseteq S\}$ such that for $\xi \in X^\omega$ it holds $\xi \in M$ if and only if $\mathrm{Inf}(\mathcal{A}; \xi) \in \mathcal{T}$ where $\mathrm{Inf}(\mathcal{A}; \xi)$ is the set of all states $s \in S$ through which the automaton $A$ runs infinitely often when reading the input $\xi$. Observe that $Z = \mathrm{Inf}(A; \xi)$ holds for a subset $Z \subseteq S$ if and only if

1. there is a word $u \in X^*$ such that $\delta(s_0; u) \in Z$, and

2. for every $s \in Z$ there is a non-empty words $v \in X^*$ such that $\delta(s, v) = s$ and $Z = \{\delta(s, v') \mid v' \sqsubseteq v\}$.

Such sets were referred to as *essential sets* [Wag79] or *loops* [SW08],[Sta97, Section 5.1]. The set of all loops of an automaton $\mathcal{A}$ will be referred to as $\mathrm{LOOP}_\mathcal{A} = \{\mathrm{Inf}(\mathcal{A}; \xi) \mid \xi \in X^\omega\}$.

Thus, to ease our notation, unless stated otherwise in the sequel we will assume all automata to be initially connected, that is, $S = \{\delta(s_0; w) \mid w \in X^*\}$.

The $\omega$-language $L(\mathcal{A},\mathcal{T}) = \{\xi \mid \mathrm{Inf}(A;\xi) \in \mathcal{T}\}$ is the (disjoint) union of all sets $M_Z = \{\xi \mid \mathrm{Inf}(A;\xi) = Z\}$ where $Z \in \mathcal{T}$. Observe that $M_Z$ and $M_{Z'}$ are disjoint for $Z \neq Z'$. Thus it holds the following.

**Lemma 20** *Let* $\mathcal{A} = (X; S; s_0; \delta)$ *be a deterministic automaton and* $\mathcal{T}, \mathcal{T}' \subseteq 2^S$ *be tables, and let* **op** *be a Boolean set operation. Then* $L(\mathcal{A},\mathcal{T})$ **op** $L(\mathcal{A},\mathcal{T}') = L(\mathcal{A},\mathcal{T}\,\mathbf{op}\,\mathcal{T}')$. *Moreover, for* $\mathcal{T}, \mathcal{T}' \in 2^S$ *we have* $L(\mathcal{A},\mathcal{T}) \subseteq L(\mathcal{A},\mathcal{T}')$ *if and only if* $\mathcal{T} \cap \mathrm{LOOP}_\mathcal{A} \subseteq \mathcal{T}' \cap \mathrm{LOOP}_\mathcal{A}$.

For $Z_1, Z_2 \subseteq S$ we write $Z_1 \mapsto Z_2$ if there exists an $s \in Z_1$ and a $w \in X^*$ such that $\delta(s,w) \in Z_2$. For $s \in S$, we write also $s \mapsto Z_2$ instead of $\{s\} \mapsto Z_2$. For simplicity we restrict $\mathcal{T}$ to $\mathcal{T} \cap \mathrm{LOOP}_\mathcal{A}$.

The relation $\mapsto$ is reflexive and transitive over $\mathrm{LOOP}_\mathcal{A}$, thus a preorder. Their maximal elements are just the *terminal* loops $\mathcal{L}_{\mathrm{term}} = \{Z \mid Z \in \mathrm{LOOP}_\mathcal{A} \wedge \forall Z'((Z \mapsto Z') \to (Z' \mapsto Z))\}$ which will be of some importance for the following considerations. Moreover we define the sets of *successor* loops $\mathcal{S}(Z) = \{Z' \mid Z' \in \mathrm{LOOP}_\mathcal{A} \wedge Z \mapsto Z'\}$. We have the following easily verified properties.

**Lemma 21** $\mathcal{S}(Z) \cap \mathcal{L}_{\mathrm{term}} \neq \emptyset$ *for all* $Z \neq \emptyset$, *and if* $Z' \in \mathcal{S}(Z)$ *then* $\mathcal{S}(Z) \supseteq \mathcal{S}(Z')$.

For a given automaton $\mathcal{A} = (X; S; s_0; \delta)$ and a table $\mathcal{T} \subseteq 2^S$ we introduce further the set of positive (negative) *successors* $\mathcal{S}_+(Z)$ ($\mathcal{S}_-(Z)$) and the set of *alternating* loops $\mathcal{S}_o$.

$$\begin{aligned} \mathcal{S}_+(Z) &=_{\mathrm{def}} \mathcal{S}(Z) \cap \mathcal{T}, \\ \mathcal{S}_-(Z) &=_{\mathrm{def}} \mathcal{S}(Z) \setminus \mathcal{T}, \\ \mathcal{S}_o &=_{\mathrm{def}} \{Z \mid \exists Z'(Z \mapsto Z' \mapsto Z \wedge (Z \in \mathcal{T} \leftrightarrow Z' \notin \mathcal{T}))\} \end{aligned} \tag{14}$$

Moreover, for $\mathcal{A} = (X; S; s_0; \delta)$ and a table $\mathcal{T} \subseteq 2^S$ we need the following terminal variants.

$$\begin{aligned} \mathcal{S}'_+(Z) &=_{\mathrm{def}} (\mathcal{S}(Z) \cap \mathcal{L}_{\mathrm{term}} \cap \mathcal{T}) \setminus \mathcal{S}_o\,, \\ \mathcal{S}'_-(Z) &=_{\mathrm{def}} ((\mathcal{S}(Z) \cap \mathcal{L}_{\mathrm{term}}) \setminus \mathcal{T}) \setminus \mathcal{S}_o\,, \text{ and} \\ \mathcal{S}'_o(Z) &=_{\mathrm{def}} \mathcal{S}(Z) \cap \mathcal{L}_{\mathrm{term}} \cap \mathcal{S}_o\,. \end{aligned} \tag{15}$$

Then the following lemma holds.

**Lemma 22** *Let* $Z \in \mathcal{L}_{\mathrm{term}}$. *Then* $Z \in \mathcal{T} \setminus \mathcal{S}_o$ *if and only if* $\mathcal{S}_-(Z) = \emptyset$.

**Proof.** Consider $Z' \in \mathcal{S}(Z)$. Then, since $Z \in \mathcal{L}_{\mathrm{term}}$, we have $Z \mapsto Z' \mapsto Z$. Consequently, $Z \notin \mathcal{S}_o$ implies $Z' \in \mathcal{T}$.

Conversely, if $\mathcal{S}_-(Z) = \emptyset$ then $Z \in \mathcal{T}$ and $Z \notin \mathcal{S}_o$. ❑

Observe that for $Z \notin \mathcal{L}_{\mathrm{term}}$ one might have $\mathcal{S}_-(Z) \neq \emptyset$ while $Z \in \mathcal{T} \setminus \mathcal{S}_o$.

**Lemma 23** *Let* $\mathcal{A} = (X; S; s_0; \delta)$ *be an automaton,* $\mathcal{T} \subseteq 2^S$ *a table and* $Z \in \mathrm{LOOP}_{\mathcal{A}}$ *and* $\delta(s_0, w) \in Z$*, for* $w \in X^*$*. Then* $\mathcal{S}_-(Z) = \emptyset$ *if and only if* $w \cdot X^\omega \subseteq L(\mathcal{A}, \mathcal{T})$.

**Proof.** Let $\xi \in w \cdot X^\omega$. Since $\delta(s_0, w) \in Z$, we have $\mathcal{S}(Z) \mapsto \mathrm{Inf}(A; \xi)$. Thus $\mathcal{S}_-(Z) = \emptyset$ implies $\mathrm{Inf}(\mathcal{A}; \xi) \in \mathcal{T}$.

Conversely, let $w \cdot X^\omega \subseteq L(\mathcal{A}, \mathcal{T})$ and $Z' \in \mathcal{S}(Z)$. Then there are words $v, u \in X^*, u \neq e$, such that $\delta(s_0; wv) = \delta(s_0; wvu) \in Z'$ and $\{\delta(s_0; wvu') \mid u' \sqsubseteq u\} = Z'$. Thus $\mathrm{Inf}(\mathcal{A}; wvu^\omega) = Z'$ which implies $Z' \in \mathcal{T}$. ❑

**Lemma 24** *Let* $\mathcal{A} = (X; S; s_0; \delta)$ *be an automaton,* $\mathcal{T} \subseteq 2^S$ *a table and* $Z \in \mathrm{LOOP}_{\mathcal{A}}$*. Then* $\mathcal{S}'_+(Z) \neq \emptyset$ *if and only if there is a* $Z' \in \mathcal{S}(Z)$ *such that* $\mathcal{S}_-(Z') = \emptyset$.

**Proof.** If $\mathcal{S}'_+(Z) \neq \emptyset$ there is a $Z' \in \mathcal{L}_{\mathrm{term}}$ such that $Z' \in \mathcal{T}$ and $Z' \notin \mathcal{S}_\circ$. According to Lemma 22, $\mathcal{S}_-(Z') = \emptyset$.

If, conversely, there is a $Z' \in \mathcal{S}(Z)$ with $\mathcal{S}_-(Z') = \emptyset$ then there is also a $Z'' \in \mathcal{S}(Z') \cap \mathcal{L}_{\mathrm{term}}$ with $\mathcal{S}_-(Z'') = \emptyset$. Again, Lemma 22 shows $\mathcal{S}(Z'') \subseteq \mathcal{T}$. ❑

Defining $C\mathcal{T} =_{\mathrm{def}} \{Z \mid Z \in \mathrm{LOOP}_{\mathcal{A}} \wedge \mathcal{S}_+(Z) \neq \emptyset\}$ and $I\mathcal{T} =_{\mathrm{def}} \{Z \mid Z \in \mathrm{LOOP}_{\mathcal{A}} \wedge \mathcal{S}_-(Z) = \emptyset\}$ we have

**Proposition 25** *Let* $\mathcal{A} = (X; S; s_0; \delta)$ *be an automaton and* $\mathcal{T} \subseteq 2^S$ *be a table. Then* $C\mathcal{T} = \mathrm{LOOP}_{\mathcal{A}} \setminus I(\mathrm{LOOP}_{\mathcal{A}} \setminus \mathcal{T})$ .

**Proof.** Let $Z \in \mathrm{LOOP}_{\mathcal{A}}$.

$Z \in C_{\mathcal{A}}\mathcal{T}$ if and only if $\mathcal{S}(Z) \cap \mathcal{T} \neq \emptyset$, and $Z \in I_{\mathcal{A}}(\mathrm{LOOP}_{\mathcal{A}} \setminus \mathcal{T})$ if and only if $\mathcal{S}(Z) \cap (\mathrm{LOOP}_{\mathcal{A}} \setminus \mathcal{T}) = \emptyset$, that is $\mathcal{S}(Z) \cap \mathcal{T} = \emptyset$. Thus $Z \in C_{\mathcal{A}}\mathcal{T}$ is equivalent to $Z \notin I_{\mathcal{A}}(\mathrm{LOOP}_{\mathcal{A}} \setminus \mathcal{T})$. ❑

Moreover, it holds the following.

**Lemma 26** *Let* $\mathcal{A} = (X; S; s_0; \delta)$ *be an automaton and* $\mathcal{T} \subseteq 2^S$ *be a table. Then*

1. $CI\mathcal{T} = \{Z \mid \mathcal{S}'_+(Z) \neq \emptyset\}$
2. $IC\mathcal{T} = \{Z \mid \mathcal{S}'_-(Z) = \emptyset\}$
3. $CIC\mathcal{T} = \{Z \mid \mathcal{S}'_+(Z) \cup \mathcal{S}'_o(Z) \neq \emptyset\}$
4. $ICI\mathcal{T} = \{Z \mid \mathcal{S}'_-(Z) \cup \mathcal{S}'_o(Z) = \emptyset\}$

**Proof.** 1. $Z \in CI\mathcal{T}$ if there is a $Z' \in \mathcal{S}(Z)$ such that $Z' \in I\mathcal{T}$, that is, $\mathcal{S}_-(Z') = \emptyset$. In view of Lemma 24 this is equivalent to $\mathcal{S}'_+(Z) \neq \emptyset$.

2. Using Proposition 25 twice, we obtain $IC\mathcal{T} = \mathrm{LOOP}_{\mathcal{A}} \setminus CI(\mathrm{LOOP}_{\mathcal{A}} \setminus \mathcal{T})$. By 1. and Eq. (15) we have $Z \in CI(\mathrm{LOOP}_{\mathcal{A}} \setminus \mathcal{T})$ if and only if $\mathcal{S}(Z) \cap$

$\mathcal{L}_{term} \cap (LOOP_{\mathcal{A}} \setminus \mathcal{T}) \setminus \mathcal{S}_o = ((\mathcal{S}(Z) \cap \mathcal{L}_{term}) \setminus \mathcal{T}) \setminus \mathcal{S}_o \neq \emptyset$. This is equivalent to $\mathcal{S}_-(Z) \neq \emptyset$.

3. We have $Z \in \mathbf{C}\mathbf{I}\mathbf{C}\mathcal{T}$ if and only if there is a $Z' \in \mathcal{S}(Z)$ such that $\mathcal{S}'_-(Z') = \emptyset$. The latter is equivalent to $\mathcal{S}'_+(Z') \cup \mathcal{S}'_o(Z') \neq \emptyset$.

4. This is completely analogous to 3. ❑

**Theorem 27** *Let* $\mathcal{A} = (X; S; s_0; \delta)$ *be an automaton and* $\mathcal{T} \subseteq 2^S$ *a table.*

1. $L(\mathcal{A}, \mathbf{I}\mathcal{T}) = \mathbf{I}L(\mathcal{A}, \mathcal{T})$
2. $L(\mathcal{A}, \mathbf{C}\mathcal{T}) = \mathbf{C}L(\mathcal{A}, \mathcal{T})$
3. $L(\mathcal{A}, \mathbf{I}\mathbf{C}\mathcal{T}) = \mathbf{I}\mathbf{C}L(\mathcal{A}, \mathcal{T})$
4. $L(\mathcal{A}, \mathbf{C}\mathbf{I}\mathcal{T}) = \mathbf{C}\mathbf{I}L(\mathcal{A}, \mathcal{T})$
5. $L(\mathcal{A}, \mathbf{I}\mathbf{C}\mathbf{I}\mathcal{T}) = \mathbf{I}\mathbf{C}\mathbf{I}L(\mathcal{A}, \mathcal{T})$
6. $L(\mathcal{A}, \mathbf{C}\mathbf{I}\mathbf{C}\mathcal{T}) = \mathbf{C}\mathbf{I}\mathbf{C}L(\mathcal{A}, \mathcal{T})$

**Proof.** Let $\mathcal{A} = (X, S, \delta, s_0)$ and $\mathcal{T} \subseteq 2^S$.

1. If $\xi \in L(\mathcal{A}, \mathbf{I}\mathcal{T})$ then $\mathcal{S}_-(\mathrm{Inf}(\mathcal{A}, \xi)) = \emptyset$ and there is a $w \sqsubset \xi$ such that $\delta(s_0; w) \in \mathrm{Inf}(\mathcal{A}, \xi)$. Now Lemma 23 shows $w \cdot X^\omega \subseteq L(\mathcal{A}, \mathcal{T})$. Thus $\xi \in \mathbf{I}L(\mathcal{A}, \mathcal{T})$.

Conversely, let $\xi \in \mathbf{I}L(\mathcal{A}, \mathcal{T})$. Then there is a $w \in X^*, w \sqsubset \xi$ such that $\delta(s_0; w) \in \mathrm{Inf}(\mathcal{A}, \xi)$ and $w \cdot X^\omega \subseteq L(\mathcal{A}, \mathcal{T})$. Again, Lemma 23 shows $\mathcal{S}_-(\mathrm{Inf}(\mathcal{A}, \xi)) = \emptyset$, that is, $\xi \in L(\mathcal{A}, \mathbf{I}\mathcal{T})$.

2. Follows from the identities $L(\mathcal{A}, LOOP_{\mathcal{A}} \setminus \mathcal{T}) = X^\omega \setminus L(\mathcal{A}, \mathcal{T})$ and $\mathbf{C}\mathcal{T} = LOOP_{\mathcal{A}} \setminus \mathbf{I}(LOOP_{\mathcal{A}} \setminus \mathcal{T})$.

3.-6. are immediate consequences of 1. and 2. ❑

Using Lemma 20 we can re-formulate the conditions $(A_0) \ldots (H)$.

**Corollary 28** *Let* $\mathcal{A} = (X; S; s_0; \delta)$ *be an automaton,* $\mathcal{T} \subseteq 2^S$ *a table, and* $M = L(\mathcal{A}, \mathcal{T})$. *Then*

| | | |
|---|---|---|
| $(A_0)$ $\mathbf{C}M = M$ | $\iff$ | $\forall Z(Z \in LOOP_{\mathcal{A}} \to (Z \in \mathcal{T} \vee \mathcal{S}_+(Z) = \emptyset))$ |
| $(A_1)$ $\mathbf{I}M = M$ | $\iff$ | $\forall Z(Z \in LOOP_{\mathcal{A}} \to (Z \notin \mathcal{T} \vee \mathcal{S}_-(Z) = \emptyset))$ |
| $(B_0)$ $\mathbf{C}\mathbf{I}\mathbf{C}M \subseteq M$ | $\iff$ | $\forall Z(Z \in LOOP_{\mathcal{A}} \to (Z \in \mathcal{T} \vee \mathcal{S}'_+(Z) = \mathcal{S}'_o(Z) = \emptyset))$ |
| $(B_1)$ $\mathbf{I}\mathbf{C}\mathbf{I}M \supseteq M$ | $\iff$ | $\forall Z(Z \in LOOP_{\mathcal{A}} \to (Z \notin \mathcal{T} \vee \mathcal{S}'_-(Z) = \mathcal{S}'_o(Z) = \emptyset))$ |
| $(C_0)$ $\mathbf{C}\mathbf{I}M \subseteq M$ | $\iff$ | $\forall Z(Z \in LOOP_{\mathcal{A}} \to (Z \in \mathcal{T} \vee \mathcal{S}'_+(Z) = \emptyset))$ |
| $(C_1)$ $\mathbf{I}\mathbf{C}M \supseteq M$ | $\iff$ | $\forall Z(Z \in LOOP_{\mathcal{A}} \to (Z \notin \mathcal{T} \vee \mathcal{S}'_-(Z) = \emptyset))$ |
| $(D_0)$ $\mathbf{I}\mathbf{C}M \subseteq M$ | $\iff$ | $\forall Z(Z \in LOOP_{\mathcal{A}} \to (Z \in \mathcal{T} \vee \mathcal{S}'_-(Z) \neq \emptyset))$ |
| $(D_1)$ $\mathbf{C}\mathbf{I}M \supseteq M$ | $\iff$ | $\forall Z(Z \in LOOP_{\mathcal{A}} \to (Z \notin \mathcal{T} \vee \mathcal{S}'_+(Z) \neq \emptyset))$ |
| $(E_0)$ $\mathbf{I}\mathbf{C}\mathbf{I}M \subseteq M$ | $\iff$ | $\forall Z(Z \in LOOP_{\mathcal{A}} \to (Z \in \mathcal{T} \vee \mathcal{S}'_-(Z) \neq \emptyset \vee \mathcal{S}'_o(Z) \neq \emptyset))$ |
| $(E_1)$ $\mathbf{C}\mathbf{I}\mathbf{C}M \supseteq M$ | $\iff$ | $\forall Z(Z \in LOOP_{\mathcal{A}} \to (Z \notin \mathcal{T} \vee \mathcal{S}'_+(Z) \neq \emptyset \vee \mathcal{S}'_o(Z) \neq \emptyset))$ |
| $(F_0)$ $\mathbf{I}\mathbf{C}\mathbf{I}M \supseteq \mathbf{C}\mathbf{I}M$ | $\iff$ | $\forall Z(Z \in LOOP_{\mathcal{A}} \to \mathcal{S}'_+(Z) = \emptyset \vee \mathcal{S}'_-(Z) = \mathcal{S}'_o(Z) = \emptyset)$ |
| $(F_1)$ $\mathbf{C}\mathbf{I}\mathbf{C}M \subseteq \mathbf{I}\mathbf{C}M$ | $\iff$ | $\forall Z(Z \in LOOP_{\mathcal{A}} \to \mathcal{S}'_-(Z) = \emptyset \vee \mathcal{S}'_+(Z) = \mathcal{S}'_o(Z) = \emptyset)$ |
| $(G)$ $\mathbf{I}\mathbf{C}M \subseteq \mathbf{C}\mathbf{I}M$ | $\iff$ | $\forall Z(Z \in LOOP_{\mathcal{A}} \to \mathcal{S}'_+(Z) \neq \emptyset \vee \mathcal{S}'_-(Z) \neq \emptyset)$ |
| $(H)$ $\mathbf{C}\mathbf{I}M \subseteq \mathbf{I}\mathbf{C}M$ | $\iff$ | $\forall Z(Z \in LOOP_{\mathcal{A}} \to \mathcal{S}'_+(Z) = \emptyset \vee \mathcal{S}'_-(Z) = \emptyset)$ |

**Proof.** Items $(A_0)$ to $(E_1)$ are proved along the following lines. E.g. for $(E_1)$, Lemma 26 and Theorem 27 yield $CICM = L(\mathcal{A}, \{Z \mid \mathcal{S}'_+(Z) \cup \mathcal{S}'_o(Z) \neq \emptyset\})$ and $M = L(\mathcal{A}, \mathcal{T})$.

Then Lemma 20 shows that $L(\mathcal{A}, CIC\mathcal{T}) \supseteq L(\mathcal{A}, \mathcal{T})$ is equivalent to $\forall Z(Z \in \mathrm{LOOP}_{\mathcal{A}} \to (\mathcal{S}'_+(Z) \cup \mathcal{S}'_o(Z) \neq \emptyset \to Z \in \mathcal{T}))$, that is, $\forall Z(Z \in \mathrm{LOOP}_{\mathcal{A}} \to (Z \notin \mathcal{T} \vee \mathcal{S}'_+(Z) \neq \emptyset \vee \mathcal{S}'_o(Z) \neq \emptyset))$.

In the case of Item $(F_0)$ we obtain in a similar way that $L(\mathcal{A}, ICI\mathcal{T}) \supseteq L(\mathcal{A}, CI\mathcal{T})$ is equivalent to $\forall Z(Z \in \mathrm{LOOP}_{\mathcal{A}} \to (\mathcal{S}'_-(Z) \cup \mathcal{S}'_o(Z) = \emptyset \to \mathcal{S}'_+(Z) \neq \emptyset))$, that is, $\forall Z(Z \in \mathrm{LOOP}_{\mathcal{A}} \to (\mathcal{S}'_-(Z) = \mathcal{S}'_o(Z) = \emptyset \vee \mathcal{S}'_+(Z) = \emptyset))$
.

The remaining items are dealt with in a similar manner. ❑

Now we look at the complexity of deciding types.

**Theorem 29** *For every type $\tau$, the problem of whether the language, accepted by a given Muller automaton, is of type $\tau$ is* NL*-complete.*

**Proof.** It is easy to see (cf. [SW08]) that, for a given automaton $\mathcal{A} = (X, S, \delta, s_0)$, a table $\mathcal{T} \subseteq 2^S$ and a set $Z \subseteq S$, the problems of whether $Z \in \mathcal{T}$, $Z \notin \mathcal{T}$, $\mathcal{S}_+(Z) = \emptyset$, $\mathcal{S}_-(Z) = \emptyset$ $\mathcal{S}'_+(Z) = \emptyset$, $\mathcal{S}'_-(Z) = \emptyset$, and $\mathcal{S}'_o(Z) = \emptyset$ are in NL (having in mind that NL is closed under complement). By Corollary 28, deciding whether a given automaton fulfils any condition $A_0, A_1, \ldots, H$ is in NL. Consequently, for any type $\tau \in \{1, \ldots, 30\}$, deciding whether a given automaton accepts a language of type $\tau$, is in NL.

For the completeness results we give reductions from the NL-complete graph accessibility problem GAP or from $\overline{\mathrm{GAP}}$ which is NL-complete, too, since the NL is closed under complement. Let $\tau \neq 30$ be a type. Choose an automaton $\mathcal{A} = (X, S, \delta, s_0)$, a table $\mathcal{T} \subseteq 2^S$ such that $L(\mathcal{A}, \mathcal{T})$ is of type $\tau$. Now consider an instance $G$ of GAP consisting of an acyclic graph $(V, E)$ such that $V \cap S = \emptyset$, a start node $s$ and a target node $t$. W.l.o.g. assume that $s$ is the only node with in-degree 0, that $t$ is of out-degree 1, and every node has an out-degree at most 2. We construct a new automaton $\mathcal{A}_G = (X, S \cup V, \delta', s, \mathcal{S})$ such that $\delta'$ works on $V$ as given by the edges of $E$, $\delta'(t, a) = s_o$, and $\delta'(v, a) = v$ for all nodes $v \neq t$ with out-degree 0. Obviously, if $G \in \mathrm{GAP}$ then $L(\mathcal{A}_G) = W \cdot L(\mathcal{A})$ for some finite set $W \subseteq X^*$, otherwise $L(\mathcal{A}_G) = \emptyset$. Since $CL(\mathcal{A}_G) = W \cdot CL(\mathcal{A})$ and $IL(\mathcal{A}_G) = W \cdot IL(\mathcal{A})$, the set $L(\mathcal{A}_G)$ is of type $\tau$ too. Hence, if $G \in \mathrm{GAP}$ then $L(\mathcal{A}_G)$ is of type $\tau \neq 30$ otherwise it is of type 30. This is a log-space reduction from GAP to the problem of whether a given automaton accepts a language of type $\tau$ and, at the same time, a log-space reduction from $\overline{\mathrm{GAP}}$ to the problem of whether a given automaton accepts a language of type 30. ❑

Finally we consider nowhere dense sets, i.e. sets $M$ such that $CICM = \emptyset$. From Lemma 26 we obtain

**Lemma 30** $L(\mathcal{A})$ *is nowhere dense if and only if* $\forall Z(\mathcal{S}'_+(Z) = \mathcal{S}'_0(Z) = \emptyset)$ .

**Theorem 31** *The problem of whether the language, accepted by a given Muller automaton, is nowhere dense is* NL*-complete.*

**Proof.** As is the proof of Theorem 29. ❑

# 3 The Topological Space of Reals

The aim of this section is to investigate which types of Kuratowski lattices exist in the space $\mathbb{R}$ of reals. This space contains only trivial sets being simultaneously open and closed. Thus it is to expect that not all of the 49 types of Kuratowski lattices exist in $\mathbb{R}$. First we consider the class of connected topological spaces to which $\mathbb{R}$ belongs.

## 3.1 Connected Spaces

As we have seen for the fulfilment of several of the types we have to require that the space $\mathcal{X}$ contains non-trivial sets being simultaneously open and closed. Connected spaces are those which contain, except for the trivial ones, $\emptyset$ and $\mathcal{X}$ itself, no other sets being simultaneously open and closed (clopen). In this section we show that indeed in connected spaces ten of the above forty-nine types are impossible.

**Theorem 32** *In a connected space there are no sets of type 12a, 12b, 13a, 13b, 14a, 14b, 18, 19a, 19b, or 27.*

**Proof.** Since there are no nontrivial clopen sets, none of the following is possible for a set $M$ from a connected space:

(a) $IM \subset ICM = CICM \subset CM$.

(b) $IM \subset ICIM = CIM \subset CM$.

(c) $IM \subset ICM = CIM \subset CM$.

Case (a) means $\neg D_0 \wedge F_0 \wedge \neg E_1$ which is fulfilled by the types 12a, 13a, 14a, 18, and 19a.

Case (b) means $\neg E_0 \wedge F_1 \wedge \neg D_1$ which is fulfilled by the types 12b, 13b, 14b, 18, and 19b.

Case (c) means $\neg D_0 \wedge G \wedge H \wedge \neg D_1$ which is fulfilled by type 27. ❑

### 3.2 All 39 Types Exist in the Space of Reals

Since the topological space of reals is connected, by Theorem 32 there are no sets of types 12a, 12b, 13a, 13b, 14a, 14b, 18, 19a, 19b, or 27. Here we show arguing in a line similar to Section 2.2 that all of the remaining 39 types exist in the space of reals.

**Proposition 33** *Let* $M \subseteq \mathbb{R}$, $a \in \mathbb{R}$, *and let* $\tau \in \{1, \ldots, 30\}$ *be any type. If* $M$ *is of type* $\tau$ *then* $M + a$ *is also of type* $\tau$.

**Proof.** This follows from $C(M + a) = CM + a$ and $I(M + a) = IM + a$. ❏

**Lemma 34** *In the topological space of real numbers, if there exist a bounded set of type* $(\alpha_1, \alpha_2, \ldots, \alpha_{10})$ *and a bounded set of type* $(\beta_1, \beta_2, \ldots, \beta_{10})$ *then there exists a bounded set of type* $(\alpha_1 \wedge \beta_1, \alpha_2 \wedge \beta_2, \ldots, \alpha_{10} \wedge \beta_{10})$.

**Proof.** Let $M_1 \subseteq \mathbb{R}$ be a bounded set of type $(\alpha_1, \alpha_2, \ldots, \alpha_{10})$, and let $M_2 \subseteq \mathbb{R}$ be a bounded set of type $(\beta_1, \beta_2, \ldots, \beta_{10})$. Since $M_1$ and $M_2$ are bounded and because of Proposition 33 we can assume that there exist a $c \in \mathbb{R}$ such that $\sup M_1 < c < \inf M_2$. Hence $I(M_1 \cup M_2) = IM_1 \cup IM_2$ and $C(M_1 \cup M_2) = CM_1 \cup CM_2$. Consequently, $M_1 \cup M_2$ fulfils a condition from $\{A_0, A_1, C_0, C_1, E_0, E_1, F_0, F_1, G, H\}$ if and only if $M_1$ and $M_2$ fulfil this condition. ❏

**Theorem 35** *In the topological space of real numbers, for every type* $\tau \notin \{12a, 12b, 13a, 13b, 14a, 14b, 18, 19a, 19b, 27\}$ *there is a set of type* $\tau$.

**Proof.** First we observe that there exist sets of types 6, 10b, 11, 16b, 17b, 20, 21, 26a, 26b, 28a, 29a, and 30 by looking at Table 6. Here $M^o =_{def} M \cap \mathbb{Q}$, for $M \subseteq \mathbb{R}$. Observe that the sets in the table, besides the one for type 20, are bounded. Thus, by Lemma 34 and Proposition 15 we obtain that also sets of types 1, 2a, 3a, 4, 5a, 5b, 7a, 8a, 9, 10a, 15b, 22a, 23a, 24a, and 25 exist. Using Proposition 6 we conclude that sets of the remaining types 2b, 3b, 5b, 7b, 8b, 15a, 16a, 17a, 22b, 23b, 24b, 28b, and 29b exist. ❏

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| type | M | CM | ICM | CICM | IM | CIM | ICIM |
|---|---|---|---|---|---|---|---|
| 6 | $(0,1)\cup(1,2)\cup(2,3)^o\cup\{4\}$ | $[0,3]\cup\{4\}$ | $(0,3)$ | $[0,3]$ | $(0,1)\cup(1,2)$ | $[0,2]$ | $(0,2)$ |
| 10b | $(0,1)^o\cup(1,2)\cup(2,3)^o$ | $[0,3]$ | $(0,3)$ | $[0,3]$ | $(1,2)$ | $[1,2]$ | $(1,2)$ |
| 11 | $(0,1)^o\cup[1,2]\cup(2,3)^o$ | $[0,3]$ | $(0,3)$ | $[0,3]$ | $(1,2)$ | $[1,2]$ | $(1,2)$ |
| 16b | $[0,1]^o$ | $[0,1]$ | $(0,1)$ | $[0,1]$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| 17b | $(0,1)^o$ | $[0,1]$ | $(0,1)$ | $[0,1]$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| 20 | $\mathbb{Q}$ | $\mathbb{R}$ | $\mathbb{R}$ | $\mathbb{R}$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| 21 | $(0,1)\cup(1,2)\cup\{3\}$ | $[0,2]\cup\{3\}$ | $(0,2)$ | $[0,2]$ | $(0,1)\cup(1,2)$ | $[0,2]$ | $(0,2)$ |
| 26a | $[0,1]$ | $[0,1]$ | $(0,1)$ | $[0,1]$ | $(0,1)$ | $[0,1]$ | $(0,1)$ |
| 26b | $(0,1)$ | $[0,1]$ | $(0,1)$ | $[0,1]$ | $(0,1)$ | $[0,1]$ | $(0,1)$ |
| 28a | $\{\frac{1}{n} \mid n \in \mathbb{N}\}$ | $\{\frac{1}{n} \mid n \in \mathbb{N}\}\cup\{0\}$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| 29a | $\{0\}$ | $\{0\}$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |
| 30 | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ | $\emptyset$ |

Table 6: Basic types in $\mathbb{R}$ (where $M^o =_{\text{def}} M \cap \mathbb{Q}$)

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