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A Polynomial-Time Algorithm for the Automatic Baire Property

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Abstract

A subset of a topological space possesses the Baire property if it can be covered by an open set up to a meagre set. For the Cantor space of infinite words Finkel introduced the automatic Baire category where both sets, the open and the meagre, can be chosen to be definable by finite automata. Here we show that, given a Muller automaton defining the original set, resulting open and meagre sets can be constructed in polynomial time.

Since the constructed sets are of simple topological structure, it is possible to construct not only Muller automata defining them but also the simpler Büchi automata. To this end we give, for a restricted class of Muller automata, a conversion to equivalent Büchi automata of at most quadratic size.

Keywords: ω-automata, Cantor space, density, Baire property

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Contents

1 Introduction

A subset of the Cantor space possesses the Baire property if it differs from an open set only by a small set, that is, by a meagre set. In the papers [Fin20, Fin21] the automatic Baire property as a special case of the ordinary Baire property was introduced, and it was shown that every automaton definable language of infinite words possesses not only the Baire property but also the automatic Baire property. To this end Finkel constructed from a given Muller automaton two Muller automata accepting the involved open and meagre set, respectively. Due to the simpler topological structure of the open and meagre sets in Finkel's definition we can characterise them also by Büchi automata.

The aim of this note is to present a polynomial time algorithm for the automatic Baire property. For any given Muller automaton $\mathcal{A}$ accepting a regular ω-language we construct in polynomial time two Muller and two Büchi automata for the required ω-languages.

Although the conversion, if possible, from Muller to Büchi automata may result in an exponential blow-up (cf. [Bok19]), we show that in our case it can be performed in polynomial time. To this end we prove that the resulting automata can be kept small, thus allowing for an application of standard polynomial-time graph algorithms.

1.1 Notation

In this section we introduce the notation used throughout the paper. By $\mathbb{N} = \{0, 1, 2, \ldots\}$ we denote the set of natural numbers. Its elements will be usually denoted by letters $i, \ldots, n$. Let X be an alphabet of cardinality $|X| = r \geqslant 2$. Then X^* is the set of finite words on X, including the *empty word* e, and X^ω is the set of infinite strings (ω-words) over X. Subsets of X^* will be referred to as *languages* and subsets of X^ω as ω-*languages*.

For $w \in X^*$ and $\eta \in X^* \cup X^\omega$ let $w \cdot \eta$ be their *concatenation*. This concatenation product extends in an obvious way to subsets $W \subseteq X^*$ and $B \subseteq X^* \cup X^\omega$. For a language W let $W^* := \bigcup_{i \in \mathbb{N}} W^i$, and $W^\omega := \{w_1 \cdots w_i \cdots : w_i \in W \setminus \{e\}\}$ be the set of infinite strings formed by concatenating non-empty words in W. Furthermore, $|w|$ is the *length* of the word $w \in X^*$ and $\mathbf{pref}(B)$ is the set of all finite prefixes of strings in $B \subseteq X^* \cup X^\omega$. We shall abbreviate $w \in \mathbf{pref}(\{\eta\})$ $(\eta \in X^* \cup X^\omega)$ by $w \sqsubseteq \eta$.

We assume the reader to be familiar with the basic facts of the theory of regular languages and finite automata. We postpone the definition of regularity for ω-languages to Section 2.1. For more details on ω-languages and regular ω-languages see the book [PP04] or the survey papers [Sta97, Tho90].

1.2 The Cantor space

We consider X^ω as a topological space (Cantor space). The *closure* (smallest closed set containing F) of a subset $F \subseteq X^\omega$, $\mathcal{C}(F)$, is described as $\mathcal{C}(F) := \{\xi : \mathbf{pref}(\{\xi\}) \subseteq \mathbf{pref}(F)\}$. The *open sets* in Cantor space are the ω-languages of the form $W \cdot X^\omega$. Countable unions of closed sets are referred to as Σ_2-*sets*, their complements as Π_2-*sets*.

Next we recall some topological notions. As usual, an ω-language $F \subseteq X^\omega$ is *dense in* X^ω if $\mathcal{C}(F) = X^\omega$. This is equivalent to $\mathbf{pref}(F) = X^*$. An ω-language $F \subseteq X^\omega$ is *nowhere dense in* X^ω if its closure $\mathcal{C}(F)$ does not contain a non-empty open subset. This property is equivalent to the fact that for all $w \in \mathbf{pref}(F)$ there is a $v \in X^*$ such that $w \cdot v \notin \mathbf{pref}(F)$. Moreover, a subset $F \subseteq X^\omega$ is *meagre* or of *first Baire category* if it is a countable union of nowhere dense sets. Meagre Π_2-sets are known to be nowhere dense (cf. [Kur66]).

2 Automata and Regularity

2.1 Regular ω-languages and finite automata

As usual we call a language $W \subseteq X^*$ *regular* if there is a finite (deterministic) automaton $A = (X; S; s_0; \delta)$, where S is the finite set of states, $s_0 \in S$ is the initial state and $\delta : S \times X \to S$ is the transition function[1], such that $W = \{w : \delta(s_0; w) \in S'\}$ for some fixed set $S' \subseteq S$.

An ω-language $F \subseteq X^\omega$ is called *regular* provided there are a finite (deterministic) automaton $A = (X; S; s_0; \delta)$ and a table $\mathcal{T} \subseteq \{Z : Z \subseteq S\}$ such that for $\xi \in X^\omega$ it holds $\xi \in F$ if and only if $\mathrm{Inf}(A; \xi) \in \mathcal{T}$ where $\mathrm{Inf}(A; \xi) := \bigcap_{w \sqsubset \xi} \{\delta(s_0, v) : w \sqsubset v \sqsubset \xi\}$ is the set of all states $s \in S$ through which the automaton A runs infinitely often when reading the input ξ. Observe that $Z = \mathrm{Inf}(A; \xi)$ holds for a subset $Z \subseteq S$ if and only if

1. there is a word $u \in X^*$ such that $\delta(s_0; u) \in Z$, and

2. for all $s, s' \in Z$ there are non-empty words $w, v \in X^*$ such that $\delta(s, w) = s'$ and $\delta(s', v) = s$.

Such sets were referred to as *essential sets* [Wag79] or *loops* [SW08],[Sta97, Section 5.1], and $\mathrm{LOOP}_{\mathcal{A}} = \{\mathrm{Inf}(\mathcal{A}; \xi) : \xi \in X^\omega\}$ is the set of all loops of an automaton $\mathcal{A}$. The ω-language $L(\mathcal{A}, \mathcal{T}) = \{\xi : \mathrm{Inf}(\mathcal{A}; \xi) \in \mathcal{T}\}$ is the union of all sets $\{\xi : \mathrm{Inf}(A; \xi) = Z\}$ where $Z \in \mathcal{T}$. The pair $(\mathcal{A}, \mathcal{T})$ is usually called a Muller automaton.

In particular, $\{\xi : \mathrm{Inf}(A; \xi) = Z\}$ and $\{\xi : \mathrm{Inf}(A; \xi) = Z'\}$ are disjoint for $Z \neq Z'$. Thus it holds the following.

Lemma 1 *Let $\mathcal{A} = (X; S; s_0; \delta)$ be a deterministic automaton and $\mathcal{T}, \mathcal{T}' \subseteq 2^S$ be tables, and let* **op** *be a Boolean set operation. Then $L(\mathcal{A}, \mathcal{T})$* **op** *$L(\mathcal{A}, \mathcal{T}') = L(\mathcal{A}, \mathcal{T}\, \mathbf{op}\, \mathcal{T}')$. Moreover, for $\mathcal{T}, \mathcal{T}' \in 2^S$ we have $L(\mathcal{A}, \mathcal{T}) \subseteq L(\mathcal{A}, \mathcal{T}')$ if and only if $\mathcal{T} \cap \mathrm{LOOP}_{\mathcal{A}} \subseteq \mathcal{T}' \cap \mathrm{LOOP}_{\mathcal{A}}$.*

We are going to represent $F = L(\mathcal{A}, \mathcal{T})$ by languages derived from the automaton $\mathcal{A}$. As in [Sta98, Sta15] refer to a word $v \in X^*, v \neq e$, as $(s; Z)$-*loop completing* if and only if

1. $\delta(s, v) = s$ and $\{\delta(s, v') : v' \sqsubseteq v\} = Z$, and

2. $\{\delta(s, v') : v' \sqsubseteq v''\} \neq Z$ for all proper prefixes $v'' \sqsubset v$ with $\delta(s, v'') = s$.

[1] We use the same symbol δ to denote the usual extension of the function δ to $S \times X^*$.

Denote by $V_{(s;Z)}$ the set of all $(s;Z)$-loop completing words, and by $U_s := \{w : \delta(s_0, w) = s\}$ the set of all words leading to the state $s \in S$. Both languages are regular and constructible from the finite automaton $\mathcal{A} = (X; S; s_0; \delta)$. Moreover, $V_{(s;Z)}$ is prefix-free and $\mathbf{pref}(V^{\omega}_{(s,Z)}) = \{w : \forall w'(w' \sqsubseteq w \to \delta(s, w') \in Z)\}$.

We obtain the following (cf. with [Sta98, Lemma 3]).

Lemma 2 *Let $\mathcal{A} = (X; S; s_0; \delta)$ be a finite automaton, and let $\mathcal{T}$ be a table. Then*

$$L(\mathcal{A}, \mathcal{T}) = \bigcup_{Z \in \mathcal{T}} \bigcup_{s \in Z} U_s \cdot V^{\omega}_{(s;Z)}. \qquad (1)$$

Thus every regular ω-language has the form $\bigcup_{j=1}^{\ell} W_j \cdot V_j^{\omega}$ where W_j, V_j are regular languages (see [Büc62, PP04, Sta97] or [Tho90]). The converse is also true, that is, if $W \subseteq X^*$ and $F, E \subseteq X^{\omega}$ are regular then also $W^{\omega}, W \cdot E$ and $E \cup F$ are regular ω-languages. Note, however, that the representation of Eq. (1) is finer, since it splits a regular ω-language $F = \bigcup_{j=1}^{\ell} W_j \cdot V_j^{\omega}$ into parts $U_s \cdot V^{\omega}_{(s;Z)}, i \in \{1, \ldots, n\}$, where, additionally, the languages $V_{(s;Z)}$ are prefix-free.

2.2 Loops, strongly connected components and density

We consider the loop structure of an automaton. For $Z_1, Z_2 \subseteq S$ we write $Z_1 \mapsto Z_2$ if $Z_1 \neq Z_2$ and there exists an $s \in Z_1$ and a $w \in X^*$ such that $\delta(s, w) \in Z_2$.

The maximal w.r.t. "$\subseteq$" loops are the *strongly connected components* $\mathrm{SCC}_{\mathcal{A}} := \{Z : Z \in \mathrm{LOOP}_{\mathcal{A}} \wedge \forall Z'(Z' \in \mathrm{LOOP}_{\mathcal{A}} \to Z' \subseteq Z \vee Z' \cap Z = \emptyset)\}$ of the automaton (multi-)graph of $\mathcal{A}$. They are the vertices of the condensation graph of $\mathcal{A}$. For vertices $Z, Z' \in \mathrm{SCC}_{\mathcal{A}}$ the relation $\mapsto$ is asymmetric and transitive, thus a partial order. Its maximal w.r.t. "$\mapsto$" elements are the *terminal strongly connected components* $\mathrm{SCC}^{t}_{\mathcal{A}} := \{Z : \forall Z'(Z' \in \mathrm{SCC}_{\mathcal{A}} \wedge Z \neq Z' \to Z \not\mapsto Z')\}$. The following properties are immediate.

Property 3 *Let $\mathcal{A} = (X; S; s_0; \delta)$ be a finite automaton and $Z \in \mathrm{SCC}_{\mathcal{A}}$.*

1. *If $z, \delta(z, w) \in Z$ then $\delta(z, w') \in Z$ for all $w' \sqsubseteq w$.*

2. *If $z \in Z$ and $Z \in \mathrm{SCC}^{t}_{\mathcal{A}}$ then $\delta(z, w) \in Z$ for all $w \in X^*$.*

We have the following relation to the density of the ω-languages $V^{\omega}_{s,Z}$.

Theorem 4 *Let $\mathcal{A} = (X; S; s_0; \delta)$ be an automaton and $Z \in \mathrm{LOOP}_{\mathcal{A}}$.*

1. *If $Z \in \mathrm{SCC}^t_{\mathcal{A}}$ then $V^{\omega}_{s,Z}$ is dense in X^{ω}, that is, $\mathcal{C}(V^{\omega}_{s,Z}) = X^{\omega}$.*
2. *If $Z \notin \mathrm{SCC}^t_{\mathcal{A}}$ then $V^{\omega}_{s,Z}$ is nowhere dense.*

Proof. First, observe that $\mathbf{pref}(V^{\omega}_{s,Z}) = \{w : \delta(s,w) \in Z\}$.

1. If $Z \in \mathrm{SCC}^t_{\mathcal{A}}$ then $\delta(s,w) \in Z$ for every $w \in X^*$. Thus $\mathbf{pref}(V^{\omega}_{s,Z}) = X^*$, that is, $\mathcal{C}(V^{\omega}_{s,Z}) = X^{\omega}$.

2. Let $w \in \mathbf{pref}(V^{\omega}_{s,Z})$ and let $\delta(s',u) \notin Z$ for some $s' \in Z$ and $u \in X^*$. Consider $v \in X^*$ such that $\delta(s_0, wv) = s'$. Since $\delta(s_0, wvu) \notin Z$ we have $wvx \notin \mathbf{pref}(V^{\omega}_{s,Z})$. □

Since $\delta(z,w) \in Z$ for $z \in Z \in \mathrm{SCC}^t_{\mathcal{A}}$, we have the following.

Lemma 5 *If $Z \in \mathrm{SCC}^t_{\mathcal{A}}$ then $\mathrm{L}(\mathcal{A}, 2^Z) = \{\xi : \mathrm{Inf}(\mathcal{A};\xi) \cap Z \neq \emptyset\} = \{w : w \in X^* \wedge \delta(s_0,w) \in Z\} \cdot X^{\omega}$ is open in X^{ω}.*

Lemma 6 *If $\mathcal{T} \cap \mathrm{SCC}^t_{\mathcal{A}} = \emptyset$ then $\mathrm{L}(\mathcal{A}, \mathcal{T})$ is of first Baire category.*

Proof. If $F \subseteq X^{\omega}$ is nowhere dense then also $w \cdot F$ is nowhere dense. Now the assertion follows with Lemma 2 and Theorem 4.2. □

2.3 The automatic Baire property

In this section we are going to prove an automatic version of the result stating that every Borel (and even every analytic) set has the Baire property.

We recall some basic definitions about meagre sets, see [Kur66, Oxt80]. An ω-language $F \subseteq X^{\omega}$ is said to be of *first Baire category* or *meagre* if it is the union of countably many nowhere dense sets, or equivalently if it is included in a countable union of closed sets with empty interiors.

Definition 1 *A subset $F \subseteq X^{\omega}$ has the* Baire property *if there is an open set $E \subseteq X^{\omega}$ such that their symmetric difference $F \,\Delta\, E$ is of first Baire category.*

An important result of descriptive set theory is the following result, see [Kur66, Oxt80].

Theorem 7 *Every Borel set of the Cantor space has the Baire property.*

In [Fin20, Fin21] an automatic version of the above theorem is proved. We first give the definition.

Definition 2 (Automatic Baire property) *A subset* $F \subseteq X^\omega$ *fulfils the* Automatic Baire property *if there are a regular and open* ω*-language* E *and a regular* ω*-language of first Baire category* F' *such that* $F \,\Delta\, E \subseteq F'$.

Then it holds the following.

Theorem 8 ([Fin20, Fin21]) *Every regular* ω*-language fulfils the Automatic Baire property.*

For the purposes of our paper we give a proof.

Proof. For a table $\mathcal{T} \subseteq 2^S$ we have

$$\begin{aligned}\mathcal{T} &= (\mathcal{T} \cap \mathrm{SCC}^t_{\mathcal{A}}) \cup (\mathcal{T} \setminus \mathrm{SCC}^t_{\mathcal{A}}) \subseteq (\mathcal{T} \cap \mathrm{SCC}^t_{\mathcal{A}}) \cup (2^S \setminus \mathrm{SCC}^t_{\mathcal{A}}) \\ &\subseteq \{2^Z : Z \in \mathcal{T} \cap \mathrm{SCC}^t_{\mathcal{A}}\} \cup (2^S \setminus \mathrm{SCC}^t_{\mathcal{A}}).\end{aligned}$$

Since $\mathcal{T} \cap \mathrm{SCC}^t_{\mathcal{A}} = \{2^Z : Z \in \mathcal{T} \cap \mathrm{SCC}^t_{\mathcal{A}}\} \cap \mathrm{SCC}^t_{\mathcal{A}}$, we have $\mathcal{T} \,\Delta\, \{2^Z : Z \in \mathcal{T} \cap \mathrm{SCC}^t_{\mathcal{A}}\} \subseteq 2^S \setminus \mathrm{SCC}^t_{\mathcal{A}}$.

Then Lemma 1 yields $L(\mathcal{A}, \mathcal{T}) \,\Delta\, L(\mathcal{A}, \{2^Z : Z \in \mathcal{T} \cap \mathrm{SCC}^t_{\mathcal{A}}\}) \subseteq L(\mathcal{A}, 2^S \setminus \mathrm{SCC}^t_{\mathcal{A}})$. According to Lemmata 5 and 6 the ω-languages $L(\mathcal{A}, \{2^Z : Z \in \mathcal{T} \cap \mathrm{SCC}^t_{\mathcal{A}}\})$ and $L(\mathcal{A}, 2^S \setminus \mathrm{SCC}^t_{\mathcal{A}})$ are open and of first Baire category, respectively. □

As a consequence we obtain that we can bound the topological complexity of the ω-language F' in Definition 2 .

Corollary 9 *If* $F \subseteq X^\omega$ *is regular then there are regular* $E, F' \subseteq X^\omega$ *such that* $F \,\Delta\, E \subseteq F'$ *where* E *is open and* F' *is a* Σ_2*-set of first Baire category.*

Since every meagre Π_2-set is already nowhere dense, and $F = \{a, b\}^* \cdot a^\omega$ is dense in $\{a, b\}^\omega$ there are no open $E \subseteq \{a, b\}^\omega$ and no Π_2-set F' such that $F \,\Delta\, E \subseteq F'$. Thus Corollary 9 cannot be improved to simpler Borel sets.

3 Transformation of Muller to Büchi Automata

In this section we consider the *Büchi acceptance* of ω-automata. An ω-language $F \subseteq X^\omega$ is Büchi accepted by an automaton $\mathcal{A} = (X; S; s_0; \delta)$ and a set of states $T \subseteq S$ if $F = \{\xi : \mathrm{Inf}(\mathcal{A}, \xi) \cap T \neq \emptyset\}$.

It is well known [Lan69] (cf. also [PP04, Sta97, Tho90] or [Wag79]) that an ω-language $F \subseteq X^\omega$ is accepted by a finite deterministic Büchi automaton if and only if for every deterministic finite automaton $\mathcal{A}$ and table $\mathcal{T}$ such that $F = L(\mathcal{A}, \mathcal{T})$ the table $\mathcal{T}$ is upwards closed, that is, $Z \in \mathcal{T} \cap \mathrm{LOOP}_{\mathcal{A}}$ and $Z' \supseteq Z, Z' \in \mathrm{LOOP}_{\mathcal{A}}$ imply $Z' \in \mathcal{T}$. This is equivalent to the fact that the ω-language $F \subseteq X^\omega$ is the complement of a regular Σ_2-set.

Whether a translation from a Muller automaton $(\mathcal{A}, \mathcal{T})$ to a Büchi automaton $(\mathcal{A}', T)$ is polynomial, is unknown (cf. [Bok19]). In case when the table $\mathcal{T}$ satisfies a certain condition we obtain a quadratic increase in the number of states. This implies that the obtained Büchi automaton has only a polynomial increase in size.

Theorem 10 *Let* $\mathcal{A} = (X; S; s_0; \delta)$ *be an automaton and* $\mathcal{T} \subseteq 2^S$ *be table all of whose loops are maximal. Then there is a deterministic Büchi automaton* $(\mathcal{A}'; T)$ *with* $O(|S|^2)$ *states such that* $L(\mathcal{A}, \mathcal{T}) = L(\mathcal{A}'; T)$.

Proof. Let $\mathcal{T} \cap \mathrm{LOOP}_{\mathcal{A}} = \{Z_1, \dots, Z_n\}$. Since all loops are maximal, they are strongly connected components of the automaton graph, thus pairwise disjoint. For the purposes of the proof we assume that $z_1^{(i)}, z_2^{(i)}, \dots, z_{\kappa_i}^{(i)}, \kappa_i = |Z_i|$, be a fixed ordering of Z_i.

Define $\hat{\mathcal{A}} = (X; \hat{S}; \hat{s}_0; \hat{\delta})$ as follows.

$$\hat{S} := (S \times \{0\}) \cup \bigcup_{i=1}^{n} (Z_i \times \{1, \dots, \kappa_i\}), \quad (2)$$

$$\hat{s}_0 := (s_0, 0), \quad (3)$$

$$T = \{(z_{\kappa_i}^{(i)}, \kappa_i) : i \in \{1, \dots, n\} \wedge z \in Z_i\}, \quad (4)$$

for $s \in S$, and $x \in X$

$$\hat{\delta}((s, 0), x) := \begin{cases} (\delta(s, x), 0), \text{ if } \delta(s, x) \notin \{z_1^{(i)} : i \in \{1, \dots, n\}\}, & (5.a) \\ (z_1^{(i)}, 1) \quad , \text{ if } \delta(s, x) = z_1^{(i)}, i \in \{1, \dots, n\}, & (5.b) \end{cases} \quad (5)$$

and for $z \in Z_i, x \in X$ and $j \geqslant 1$

$$\hat{\delta}((z, j), x) := \begin{cases} (\delta(z, x), 0) \quad , \text{ if } \delta(z, x) \notin Z_i \vee j = \kappa_i, & (6.a) \\ (z_{j+1}^{(i)}, j + 1), \text{ if } j < \kappa_i \wedge \delta(z, x) = z_{j+1}^{(i)}, \text{ and,} & (6.b) \\ (\delta(z, x), j) \quad , \text{ otherwise.,} & (6.c) \end{cases} \quad (6)$$

Then $\hat{\delta}((z, j), x) = (\delta(s, x), j')$ with $j' \in \{0, j, j + 1\}$, and, consequently, $\hat{\delta}((z, j), w) = (\delta(s, w), j'')$ for $w \in X^*$ and some j''.

Moreover the following items hold.

1. if $z, \delta(z, x) \in Z_i$ and $j < \kappa_i$ then $\hat{\delta}((z, j), x) = (\delta(s, x), j')$ where $j' \in \{j, j + 1\}$, and

2. if $\hat{\delta}((s, j), x) = (s', j + 1)$ then $s' = \delta(s, x) = z_{j+1}^{(i)}$ for some $i \in \{1, \dots, n\}$.

This implies

3. If $j \geqslant 1$, $\hat{\delta}((s,0),w) = (\delta(s,w),j)$ and $\delta(s,w) \in Z_i$ then $\{\delta(s,w') : w' \sqsubseteq w\} \supseteq \{z_1^{(i)},\ldots,z_j^{(i)}\}$.

If $\mathrm{Inf}(\hat{\mathcal{A}};\xi) \cap T \neq \emptyset$ then there are infinitely many prefixes $u \sqsubset \xi$ such that $\hat{\delta}((s_0,0),u) = (z_{\kappa_i}^{(i)},\kappa_i)$ for some i. From Item 3 above and $\hat{\delta}((z_{\kappa_i}^{(i)},\kappa_i),x) = (\delta(z_{\kappa_i},x),0)$ we obtain infinitely many $u_\ell, w_\ell \in X^*$ and $x_\ell \in X$ such that $u_\ell \cdot x_\ell \cdot w_\ell \sqsubset u_{\ell+1} \sqsubset \xi$ where $\hat{\delta}((s_0,0),u_\ell) = (z_{\kappa_i}^{(i)},\kappa_i)$, $\hat{\delta}((s_0,0),u_\ell \cdot x_\ell) = (\delta(z_{\kappa_i}^{(i)},x_\ell),0)$ and $\{\delta(z_{\kappa_i}^{(i)},x_\ell \cdot w) : w \sqsubseteq w_\ell\} \supseteq Z_i$.

Since Z_i is a strongly connected component, $\mathrm{Inf}(\mathcal{A};\xi) = Z_i$ and $\xi \in L(\mathcal{A},\mathcal{T})$ follow.

Let now $\mathrm{Inf}(\hat{\mathcal{A}};\xi) \cap T = \emptyset$, and assume $\mathrm{Inf}(\mathcal{A};\xi) = Z_i$ for some i. Since $\mathrm{Inf}(\hat{\mathcal{A}};\xi) \cap T = \emptyset$, there is a $u \sqsubset \xi$ such that $\{\delta(s_0,w) : u \sqsubseteq w \sqsubset \xi\} \subseteq Z_i$ and $\{\hat{\delta}((s_0,0),w) : u \sqsubseteq w \sqsubset \xi\} \cap T = \emptyset$. Then for all $\hat{\delta}((s_0,0),w) = (\delta(s_0,w),j_w), u \sqsubseteq w \sqsubset \xi$, we have $j_w < \kappa_i$. Let $\hat{j} := \max\{j_w : u \sqsubseteq w \sqsubset \xi\}$. Fix $\hat{w}, u \sqsubseteq \hat{w} \sqsubset \xi$ such that $\hat{\delta}((s_0,0),\hat{w}) = (\delta(s_0,\hat{w}),\hat{j})$. Then Item 1 implies $\hat{\delta}((s_0,0),w) = (\delta(s_0,w),\hat{j})$ for all $w, \hat{w} \sqsubseteq w \sqsubset \xi$. Thus, in view of Item 2, $z_{\hat{j}+1}^{(i)} \notin \{\hat{\delta}((s_0,0),w) : \hat{w} \sqsubseteq w \sqsubset \xi\}$ and, consequently, $z_{\hat{j}+1}^{(i)} \notin Z_i$ which contradicts $\mathrm{Inf}(\mathcal{A};\xi) = Z_i$. □

4 Algorithms

After the prerequisites derived in the preceding sections we present the polynomial constructions. To this end we refer to classical graph algorithms (cf. [CLR09, BG00]) and to the results of [SW08]. It is well known that the strongly connected components of a graph can be estimated in polynomial time of the size of the graph. Similarly, its condensation graph, that is, the graph having as vertices the strongly connected components and as edges the connections induced by the underlying graph, can be also constructed in polynomial time. This allows also to find the terminal strongly connected components $\mathrm{SCC}^t_{\mathcal{A}}$ in polynomial time.

Moreover we have the following.

Lemma 11 [SW08] *Let $\mathcal{A} = (X,S,s_0,\delta)$ be deterministic automaton. Then the predicates $Z \subseteq Z'$ and $Z \in \mathrm{LOOP}_{\mathcal{A}}$, $Z,Z' \subseteq S$ are decidable in nondeterministic logarithmic space.*

4.1 Muller automata for the automatic Baire property

Assume we are given an automaton $\mathcal{A} = (X,S,s_0,\delta)$ and a table $\mathcal{T} \subseteq 2^S$ such that $F = L(\mathcal{A},\mathcal{T})$. We show that the equation $L(\mathcal{A},\mathcal{T}) \,\Delta\, L(\mathcal{A}_1,\mathcal{T}_1) \subseteq$

$X^\omega \setminus L(\mathcal{A}_2, \mathcal{T}_2)$ is satisfied for automata $\mathcal{A}_1, \mathcal{A}_2$ and tables $\mathcal{T}_1, \mathcal{T}_2$ which can be constructed in polynomial time from $\mathcal{A}$ and $\mathcal{T}$.

According to the proof of Theorem 8 and Lemma 1 we have

$$L(\mathcal{A}, \mathcal{T}) \,\Delta\, L(\mathcal{A}, \{2^Z : Z \in \mathcal{T} \cap \mathrm{SCC}^t_{\mathcal{A}}\}) \subseteq X^\omega \setminus L(\mathcal{A}, \mathrm{SCC}^t_{\mathcal{A}})\,. \tag{7}$$

Then $\mathcal{A}_2 := \mathcal{A}$ and $\mathcal{T}_2 := \mathrm{SCC}^t_{\mathcal{A}}$ are constructible from $\mathcal{A}$ in polynomial time. We have here to construct the complement of the first Baire category set $L(\mathcal{A}, 2^S \setminus \mathrm{SCC}^t_{\mathcal{A}})$ in order to avoid a possible exponential blow-up (cf. [Bok19]) in the size of the table $(2^S \setminus \mathrm{SCC}^t_{\mathcal{A}}) \cap \mathrm{LOOP}_{\mathcal{A}}$.

For the construction of $\mathcal{A}_1$ and $\mathcal{T}_1$ we use Property 3.2 which allows us to merge the elements of $Z \in \mathrm{SCC}^t_{\mathcal{A}}$ into a single state which will be denoted by Z. Thus $\mathcal{A}_1 = (X, S', s'_0, \delta')$ where

$$\begin{aligned}
S' &:= \left(S \setminus \bigcup\{Z : Z \in \mathrm{SCC}^t_{\mathcal{A}}\}\right) \cup \{s_0\} \cup \mathrm{SCC}^t_{\mathcal{A}}, \\
s'_0 &:= s_0, \\
\delta'(s,x) &:= \begin{cases} \delta(s,x), \text{ if } \delta(s,x) \in S \setminus \bigcup\{Z : Z \in \mathrm{SCC}^t_{\mathcal{A}}\}, \\ Z \quad\quad, \text{ if } Z \in \mathrm{SCC}^t_{\mathcal{A}} \text{ and } \delta(s,x) \in Z \text{ or } s = Z. \end{cases}
\end{aligned}$$

and $\mathcal{T}_1 := \{\{Z\} : Z \in \mathcal{T} \cap \mathrm{SCC}^t_{\mathcal{A}}\}$.

Then the automaton $\mathcal{A}_1$ is also constructible from $\mathcal{A}$ in polynomial time. For the table $\mathcal{T}_1$ we use Lemma 11 to select the elements of $\mathrm{SCC}^t_{\mathcal{A}}$ from the table $\mathcal{T}$. Since each comparison of $Z \in \mathrm{SCC}^t_{\mathcal{A}}$ with $Z' \in \mathcal{T}$ can be done in nondeterministic logarithmic space, it follows that the construction of $\mathcal{T}_1$ can be done in polynomial time of $\mathrm{size}(\mathcal{A}) + \mathrm{size}(\mathcal{T})$.

4.2 Büchi automata for the automatic Baire property

In order to construct the Büchi automata $(\mathcal{B}_1, T_1)$ and $(\mathcal{B}_2, T_2)$ satisfying $L(\mathcal{A}, \mathcal{T}) \,\Delta\, L(\mathcal{A}, \{2^Z : Z \in \mathcal{T} \cap \mathrm{SCC}^t_{\mathcal{A}}\}) \subseteq X^\omega \setminus L(\mathcal{A}, \mathrm{SCC}^t_{\mathcal{A}})$ we use the Muller automata constructed in the preceding section. For $(\mathcal{A}_1, \mathcal{T}_1)$ the transformation to $(\mathcal{B}_1, T_1)$ is straightforward. Set $\mathcal{B}_1 := \mathcal{A}_1$ and $T_1 := \mathcal{T} \cap \mathrm{SCC}^t_{\mathcal{A}}$. This results in a so-called weak Büchi automaton $(\mathcal{B}, T)$ where either $Z \subseteq T$ or $Z \cap T = \emptyset$ for $Z \in \mathrm{LOOP}_{\mathcal{B}}$.

The table $\mathcal{T}_2 = \mathrm{SCC}^t_{\mathcal{A}}$ of the Muller automaton $(\mathcal{A}_2, \mathcal{T}_2)$ satisfies the hypothesis of Theorem 10. Thus we can construct the Büchi automaton $(\mathcal{B}_2, T_2)$ according to the construction in the proof. Since the size of $(\mathcal{B}_2, T_2)$ is only quadratic in the number of states of $\mathcal{A}_2$ this can be done in polynomial time.

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