COMPSCI 320SC 2023 Midterm Test

Put the answers in the space below the questions. Write clearly and *show all your work*!

Marks for each question are shown below and just before each answer area.

This 60 minute test is worth 10% of your final grade for the course.

Question #:	1	2	3	4	Total
Possible marks:	5	5	5	5	20
Awarded marks:					

1. (a) Write a formal definition for the big-Oh notation, $f(n) = O(g(n))$. ***(2 marks)***

There exist a constant $c > 0$ and constant $n_0 \geq 0$, such that for all $n > n_0$, we have $f(n) \leq c \cdot g(n)$.

(b) Using part (a), show that if $a(n)$ and $b(n)$ represent the running times of two algorithms, then $\log\left(a(n)\sqrt{b(n)}\right) = O\left(\log\sqrt{a(n)} + \log b(n)\right)$. ***(3 marks)***

For all $n > 0$ with $c = 2$, we have:

$$\log\left(a(n)\sqrt{b(n)}\right) = \log a(n) + \frac{1}{2}\log b(n) \leq 2\cdot\left(\frac{1}{2}\log a(n) + \log b(n)\right) = 2\cdot\left(\log\sqrt{a(n)} + \log b(n)\right)$$

2. Consider the following preferences for the Stable Matching Problem.

Guy	Ladies			
Alex	Wendy	Xena	Yvone	Zoe
Bob	Xena	Wendy	Yvone	Zoe
Cris	Wendy	Zoe	Yvone	Xena
Dak	Yvone	Wendy	Xena	Zoe

Lady	Guys			
Wendy	Cris	Bob	Alex	Dak
Xena	Dak	Bob	Alex	Cris
Yvone	Dak	Alex	Bob	Cris
Zoe	Dak	Alex	Cris	Bob

Consider the Gale-Shapely algorithm presented in class, if there is a choice, please take lexicographic smallest named person to do the proposing.

(a) Show the sequence of proposals and the final matching when the guys do the proposing. ***(2 marks)***

A to W, B to X, C to W, W reject A, A to X, X reject A, A to Y, D to Y, Y reject A, A to Z

A=Z, B=X, C=W, D=Y

(b) Show the sequence of proposals and the final matching when the ladies do the proposing. ***(2 marks)***

W to C, X to D, Y to D, D reject X, X to B, Z to D, D reject Z, Z to A

A=Z, B=X, C=W, D=Y

(c) For the matching Alex=Wendy, Bob=Yvone, Cris=Zoe, Dak=Xena list all unstable pairs, if any. ***(1 mark)***

B-W, C-W, and D-Y

3. Consider the performer Magic Pants who performs daily tricks at the University of Auckland. He wants to do a very special trick as few times as possible so that every student can enjoy it. We have n students that attend campus on a given day with arrival times a_i and departure times d_i, with $a_i \leq d_i$ for students $1 \leq i \leq n$.

   (a) In the spirit of your Assignment 2, model (state precisely) this problem as a closed-interval problem. ***(2 marks)***

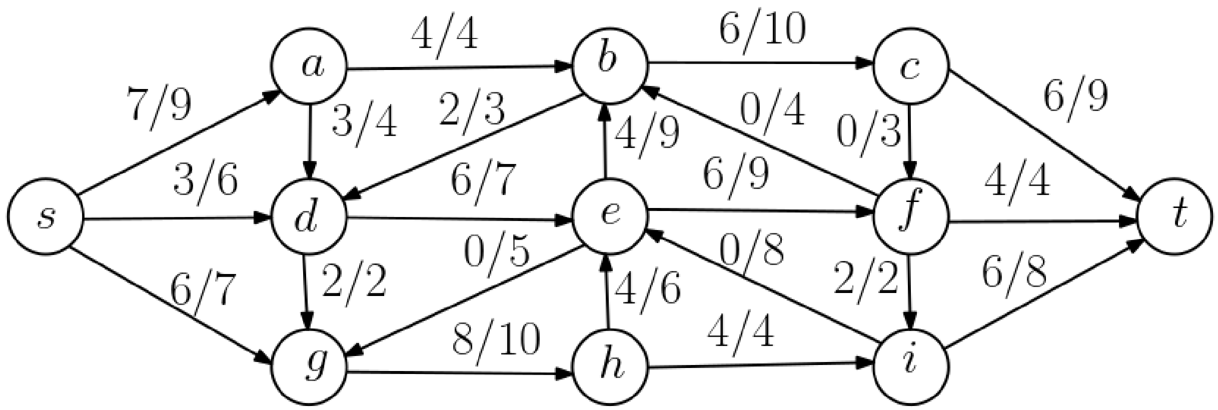
Input: $n \geq 1$ closed intervals $I_i = [a_i, d_i]$, for $1 \leq i \leq n$.
Question: We want a minimum set of m cuts $\{c_1, c_2, \ldots, c_m\}$ such that for each interval I_i we have $c_j \in I_i$ for some $1 \leq j \leq m$.

   (b) Describe a greedy algorithm that solves this problem in time $O(n \log n)$ and justify why it is correct. ***(3 marks)***

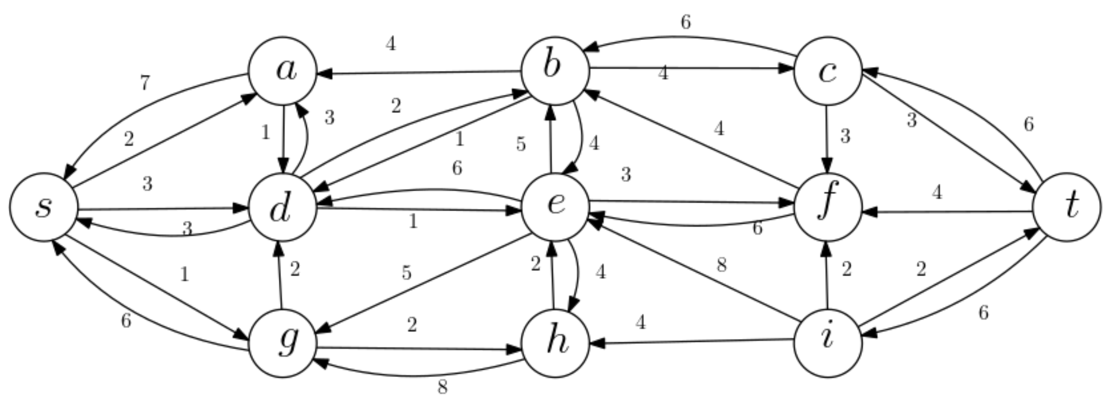
Consider the following algorithm.
Sort intervals by departure times d_i and put in list D; $d = 1$.
Boolean seen = [false, false, ...]
Set cuts $C = \emptyset$.
While not all seen:
    Set $C = C \cup \{d_d\}$
    Set seen[i]=true for all $s_i \leq d_d$ and $d_i \geq d_d$
    Set d = next index i in D with seen[i]==false
Return C

The algorithm is basically like the interval scheduling or the gas refueling algorithm, which finds the next future departure time for an interval not already cut and adds that. Hence all intervals are cut. If an optimal solution has a cut that is before d_d then we can replace it with greedy's choice with same number of cuts by moving the optimal value right to match. Note an efficient implementation would use a priority queue sorted by D containing the pair (a_i, d_i) then we skip if a_i is less or equal to the last cut added to C—thus avoid explicitly keeping a seen array.

4. Consider the following $s - t$ network with "partial flow / capacities" listed.



(a) Draw the residual digraph. ***(2 marks)***



(b) Find an augmenting path with the largest bottleneck available. ***(1 mark)***

Answer: There are a few possible augmenting paths with maximum bottleneck 2. For example, the path (s, d, b, c, t)

(c) Compute the maximum flow and give a minimum cut as a certificate. ***(2 marks)***

Answer: A minimum cut is between nodes $\{s, a, b, c, d, e, f, g, h\}$ and $\{i, t\}$ of weight $9+4+4+2 = 19$.