Computer Science 314
Notes on Logarithms and Units

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1 Introduction

It is obvious from student questions and the answers to test questions that many students know very little about logarithms. Even questions which I have designed to be trivial are made difficult because they assume knowledge which is so often absent.

This note is intended to tell you something about logarithms in an informal way with little or no proof.

2 How logarithms arise.

Logarithms arise or are used in three contexts (perhaps more!) and aspects which are important or natural in one area may be quite unimportant or unnatural in another application. As mathematical theory and understanding has developed and as calculators have become widespread, the whole emphasis of logarithms has changed. Aspects which were once of prime importance may now be quite secondary, and this may be the root of many of our problems.

2.1 As a computational tool

This is how logarithms were introduced in pre-calculator days, as an aid to multiplication, division, raising to powers and extraction of roots. The logarithm was a mysterious number which we found from looking up a table in a "log book" (which also contained sine, cosine and other interesting functions).

The properties used in calculation were —

$$\log(a \times b) = \log a + \log b \tag{1}$$
$$\log(a/b) = \log a - \log b \tag{2}$$
$$\log a^b = b \times \log a \tag{3}$$
$$\log(\sqrt[b]{a}) = (\log a)/b \tag{4}$$

The tables were headed "Common Logarithms", but the books included other rather mysterious tables of "Natural Logarithms".

Even though logarithms are now seldom used in calculations, the four equations above are essential for a lot of later work using logarithms.

2.2 With exponents and the exponential function

Eventually we progressed from simple algebra to calculus and learnt more about exponents and the exponential function. In particular, if

$$\begin{aligned} x &= y^z & (5) \\ \text{then} \quad z &= \log_y x & (6) \end{aligned}$$

(which was read as z is the "logarithm of x to base y".)

We now found that the "common logarithms" were actually logarithms to base 10 ($\log_{10}$, often just written as "log" or as "lg"), and that the "natural logarithms", often written as "ln" were logarithms to the base e, where $e = 2.7182818285\ldots$, from solving the equations $\frac{d}{dx}f(x) = f(x)$, with the solution $f(x) = e^x$.

The approach through the exponential function is the most natural for most physical systems. Physical models and descriptions are often based on linear differential equations and have solutions in exponential and similar functions. As finding the "value whose exponential is x" means finding $\ln x$, the logarithmic functions also arise naturally in these applications. The main operations with logarithms follow directly from their interpretation as powers of the base.

2.3 As the result of an integral

When studying integration we very early learn that

$$\int x^n dx = \frac{1}{n+1}x^{n+1} + C \qquad (7)$$

which clearly fails for $n = -1$. But eventually (after learning a lot more calculus) we find that

$$\ln x = \int_1^x \frac{1}{t}dt \qquad (8)$$

Much of the mathematical study of the logarithms follows from this definition, but it seems to have little direct relevance to most physical applications of logarithms (which includes data communications).

3 General properties

Given that logarithms may use *any* convenient base, it is often necessary to convert logarithms in one base into logarithms in another base. We write the standard exponential equation (equation 5 above), restate it to use logarithms and then take logs of the original equation in another base.

$$
\begin{aligned}
\text{if} \qquad a &= b^c \\
\text{then} \qquad c &= \log_b a \qquad && \text{log to base } b \\
\text{and} \qquad \log_x a &= c \log_x b \qquad && \text{log to base } x \\
&= \log_b a \log_x b \\
\text{whence} \qquad log_b a &= \frac{\log_x a}{\log_x b} && (9)
\end{aligned}
$$

In communications and related topics such as information theory we often need logarithms to base 2. The conversion is easily done using Equation 9 and either common or natural logarithms (both of which are available on most scientific calculators).

$$
\begin{aligned}
\log_2 x &= \frac{\log_{10} x}{\log_{10} 2} && (10) \\
\text{or} &= \frac{\ln x}{\ln 2} && (11)
\end{aligned}
$$

Some relations involving logarithms must be stated here. Most follow from equations 1 to 4 and some are even those equations restated. Except where a base is stated, they apply for *any* base

$$
\begin{aligned}
\log 1 &= 0 && (12) \\
\log_z z &= 1 && (13) \\
\log_z z^2 &= 2 && (14) \\
\log_z z^n &= n && (15) \\
\log_z 1/z &= -1 && (16) \\
\log 1/y &= -\log y && (17)
\end{aligned}
$$

4 Some specific examples

4.1 Signal to noise ratios

While many physical measurements involve actual units such as megahertz (MHz), microseconds (μs) or millivolts (mV), others are much more subjective ("Is this sound twice as loud as that?", "Is this light as bright as that one?"). For measurements which *compare* two things, using terms like "twice" and "a tenth" it is often best to use the logarithm of the *ratio* of the values rather than absolute values.

A good example of a ratio is the "signal to noise ratio" or "S/N ratio" where a signal is received in the presence of noise. There is a long tradition in audio engineering and acoustics of taking the logarithm of the ratio of the signal power to the noise power as the measure on "noisiness" or quality of the signal. Taking the logarithm to base 10 yields the unit of "bel" (named after Alexander Graham Bell who not only invented the telephone but did a lot of pioneering work in acoustics). As the bel is a rather too large unit for convenience, we normally use the "deciBel" or one tenth of a bel as the working unit, abbreviated as "dB".

If two signals are received with powers P_1 and P_2, their powers are in the ratio

$$\text{power ratio} = 10 \log_{10} \frac{P_1}{P_2} \text{ dB} \tag{18}$$

If a signal is received with a power S in the presence of noise with power N, the "signal-to-noise ratio" is given by

$$\text{S/N ratio} = 10 \log_{10} \frac{S}{N} \text{ dB} \tag{19}$$

Some important points and important values are –

- A *power* ratio of 10:1 corresponds to 10dB; a *voltage* (or current) ratio of 10:1 is 20dB[1].
- A power change of 1dB is barely perceptible to most people.
- A 2:1 power ratio gives $10 \log 2 = 2 \times 0.3010 \ldots \approx 3$dB.
- A signal-to-noise ratio of 0dB *does not* mean zero signal. It means that $\log(S/N) = 0$, that $S/N = 1$ or $S = N$. With appropriate design, some communication systems operate with S/N ratios of 0dB, or even -10dB (signal power one tenth of the noise power).

4.2 An example

(Taken from the 2000 314FC Test)

> According to Shannon's result the capacity C of a noisy channel with bandwidth W and signal and noise powers S and N is
>
> $$C = W \log_2(1 + S/N)$$
>
> Assuming a channel with a bandwidth of 1 MHz, calculate the channel capacities
>
> 1. with a signal noise ratio of 0 dB (zero decibels)

[1] In electronics we usually observe *voltage* ratios rather than *power* ratios. Because power is proportional to the *square* of the voltage, a 2:1 *voltage* ratio gives a 4:1 *power* ratio, or 6dB. A 10:1 voltage ratio gives a 100:1 power ratio, or 20dB.

2. with a signal noise ratio of 5 dB (assume $\log_{10} 3 \approx 0.5$)

Three points must be made here –

1. As we are working with logarithms (and to base 2) we need to understand something about logarithms.
2. We are told to "assume $\log_{10} 3 \approx 0.5$". Given that we are working with numbers that are seldom simple, a direction such as this is a very strong hint that it may simplify things.
3. Given that there is an overall instruction that calculators may not be used it is probable that the values are cunningly chosen to simplify calculations. (This fits in with the previous point.) If you find difficult numbers you have probably made a mistake.

   In data communications data rates and speeds, the prefix "kilo" *always* means $10^3 = 1000$ and *never* $2^{10} = 1024$. Similarly "mega" is always $10^6 = 1,000,000$, rather than $2^{20} = 1,048,576$. Data *quantities*, such as "megabyte", are delightfully ambiguous and can use either the decimal or the binary meaning, or even a mixture of the two!

   If you are to transmit a file of x megabytes at y megabits per second, you really need to ask *exactly* what is meant by "megabyte".

To solve the two questions –

1. A signal:noise ratio $S/N = 0$dB means that $10\log_{10}(S/N) = 0$. Therefore $(S/N) = 10^0 = 1$; the signal and noise powers are equal. Inserting into Shannon's formula

$$\begin{aligned} C &= W\log_2(1 + S/N) \\ &= W\log_2(1+1) \\ &= W\log_2(2) \\ &= W \qquad\qquad \text{because } \log_x x \equiv 1 \\ &= 1 \text{ Mbps} \end{aligned}$$

   You need to know something about logarithms, but there is no difficult calculation.

2. The signal-to-noise ratio is given as 5dB. Therefore $10\log_{10}(S/N) = 5$, or $\log_{10}(S/N) = 0.5$. But we are told to assume that $\log_{10} 3 \approx 0.5$, from which $S/N = 3$. Inserting into Shannon's formula

$$\begin{aligned} C &= W\log_2(1 + S/N) \\ &= W\log_2(1+3) \end{aligned}$$

value	comment	formula	result	correct
1	should be obvious!	—	0	0.0000
10		—	1	1.0000
100		—	2	2.0000
1000		—	3	3.0000
2	$2^{10} \approx 10^3$. $(1024 \approx 1000)$	As $\log 2 = \frac{\log x}{\log_2 x}$, $\log 2 = \frac{\log 1000}{\log_2 1000} = \frac{3}{10}$	0.3	0.3010
4	$4 = 2^2$	$\log 4 = 2 \log 2$	0.6	0.6021
8	$8 = 2^3$	$\log 8 = 3 \log 2$	0.9	0.9031
5	$5 = 10 \div 2$	$\log 5 = \log 10 - \log 2$ $= 1 - 0.3$	0.7	0.6990
9	$9^2 = 81 \approx 80$. $\log 80 = \log(10 \times 8)$ $= \log 10 + \log 8$	$\log 10 = 1, \quad \log 8 = 0.9$ $\log 9 = \frac{1+0.9}{2}$	0.95	0.9542
3	$3 = \sqrt{9}$	$\log 3 = \frac{\log 9}{2} = \frac{0.95}{2}$	0.475	0.4771
6	As $6 = 2 \times 3$, $\log 6 = \log 2 + \log 3$	$0.3 + 0.475$	0.775	0.7782
7	Whoever wants $\log 7$? But $7^2 = 49 \approx 50$	$\log 50 = \log 10 + \log 5$ $\log 7 = \frac{1.7}{2} = 0.85$	0.85	0.8451

Table 1: Derivation of Base-10 logarithms 1–10

$$
\begin{aligned}
&= W \log_2(4) \\
&= 2W \qquad \text{because } \log_x x^2 \equiv 2 \\
&= 2 \text{ Mbps}
\end{aligned}
$$

The calculation is again simple, provided that you have a knowledge of logarithms.

5 Estimating some simple logarithms.

You often need logarithms of some simple numbers. Surely you don't have to remember them all if your calculator has flat batteries?

You don't. This section and Table 1 show how to derive reasonable approximations to some frequent logarithms to base 10. Logarithms to base 2 are then these values divided by 0.3, or even multiplied by 3. And remember that a lot of the time in real-world work with signal/noise ratios, buffer sizes and so on, an error of 10% or even 20% does not matter too much to get a good *idea* of the resultant value.

Unless otherwise stated, all logarithms in this table are to base 10.

6 Units and Multipliers

Another area where a lot of answers had trouble was in the units. A bandwidth of 1 MHz (megahertz = million per second) probably translates into

name	abbrev	multiplier	importance
yotta	Y	10^{24}	v. slight
zetta	Z	10^{21}	v. slight
exa	E	10^{18}	slight
peta	P	10^{15}	slight
tera	T	10^{12}	moderate
giga	G	10^{9}	moderate
mega	M	10^{6}	very
kilo	k	10^{3}	very
milli	m	10^{-3}	very
micro	μ	10^{-6}	very
nano	n	10^{-9}	moderate
pico	p	10^{-12}	moderate
femto	n	10^{-15}	slight
atto	a	10^{-18}	slight
zepto	z	10^{-21}	v. slight
yocto	y	10^{-24}	v. slight

Table 2: Official SI (International) prefixes

megabits per second in the capacity. The logarithm itself is dimensionless; it may scale values but not otherwise change the units.

The multiplying prefixes, shown in Table 2 with their names, abbreviations and multiplying factors, are also important. The values from "tera" to "nano" are important, the extreme ones less so (but may become more widespread). Many students confused "kilo" with "mega".

In searching for some examples of **BIG** values, I found that in astronomy, 1 light year = 9.5Pm, and 1 megaparsec = 30.9 Zm.

6.1 Relations between frequency and time

You must be familiar with what could be called "reciprocal prefixes", such as {μ and M} (10^{-6} and 10^{6}), {n and G} (10^{-9} and 10^{9}) and {m and k} (10^{-3} and 10^{3}) particularly for times (say ns or μs) and frequencies (GHz and MHz).

But equally important are pairs of units, displaced so that both the period and corresponding frequency are numerically $1 \leq$ value < 1000. (Often both may be approximated to integers.) Thus it is better to say that a signal with a *frequency* of 40 MHz has a *period* of 25 ns, than to use the equivalent 0.025μs. The conversions between {MHz $\Leftrightarrow$ ns }, {GHz $\Leftrightarrow$ ps } and {kHz $\Leftrightarrow$ μs } should almost automatic.