Solutions for Assignment on Automata

(10 marks for each exercise)

Question 1.

(a)

(b)

(c)

(d)

Question 2.

DFA for union

DFA for intersection

Question 3. Let $\mathcal{A} = (S, I, T, F)$ be NFA that recognizes W. Construct NFA $\mathcal{B}$ that recognizes *Prefix*(W). Let $\mathcal{B} = (S, I, T, F')$, that is the set of states, initial states, and the transitions in $\mathcal{B}$ are the same as in $\mathcal{A}$. Let

$$F' = \{s \mid \text{there is } f \in F \text{ and there is a path from } s \text{ to } f \text{ in the transition diagram for } \mathcal{A}\},$$

in other words, F' is the set of states from which one can reach a state in F. Now it is easy to see that $\mathcal{B}$ recognizes *Prefix*(W).

Question 4. Suppose that $L = \{a^n b^n c^n \mid n \in \omega\}$ is recognizable by NFA $\mathcal{A}$ and $\mathcal{A}$ has k states. Consider $w = a^k b^k c^k$. Due to the pumping lemma, there are strings x, y, and z such that $y \neq \lambda$, $w = xyz$, and $xy^i z \in L$ for every $i \geqslant 0$. There are several cases: 1) y

contains only a's, 2) y contains both a's and b's, 3) y contains only b's, 4) y contains both b's and c's, 5) y contains only c's. In any of these cases $xy^2z \notin L$. This contradiction shows that L is not FA recognizable.

Question 5. Suppose that $L = \{ww \mid w \in \{a, b\}^*\}$ is recognizable by NFA $\mathcal{A}$ and $\mathcal{A}$ has k states. Consider $w = a^k b^k a^k b^k$ and let $p = q_0, q_1, \ldots, q_k, q_{k+1}, \ldots, q_{4k}$ be an accepting run of $\mathcal{A}$ on w. Then there exist $i < j \leqslant k$ such that $q_i = q_j$. This means that there are strings x, y, and z such that $w = xyz$, $xy^iz \in L$ for every $i \geqslant 0$, and y is a substring of the first a^k. As one can see $xy^2z \notin L$. This contradiction shows that L is not FA recognizable.

Question 6.

(a) Let $L = \{a^n, a^{n+p}, a^{n+2p}, a^{n+3p}, \ldots\}$ be an ultimately periodic language. Then L is recognizable by the following NFA $\mathcal{A} = (S, \{q_0\}, T, \{q_n\})$, where the set of states is $S = \{q_0, \ldots, q_n, r_1, \ldots, r_{p-1}\}$, q_0 is the only initial state and $\{q_n\}$ is the only final state. Note that if $p \leqslant 1$ then $S = \{q_0, \ldots, q_n\}$, and if $n = 0$ then q_0 is both initial and final state. The transition function T is defined as follows:

$$T(q_i) = \begin{cases} \{q_{i+1}\} & \text{if } i < n, \\ \{r_1\} & \text{if } i = n \text{ and } p > 1, \\ \{q_n\} & \text{if } i = n \text{ and } p = 1, \\ \varnothing & \text{if } i = n \text{ and } p = 0. \end{cases} \qquad \text{and} \qquad T(r_i) = \begin{cases} \{r_{i+1}\} & \text{if } i < p-1, \\ \{q_n\} & \text{if } i = p-1. \end{cases}$$

(b) Let L be FA recognizable language and $\mathcal{A} = (S, q_0, T, F)$ be a DFA that recognizes L. Since $\Sigma = \{a\}$ every state $s \in S$ has exactly one outgoing edge labeled with a in the transition diagram of $\mathcal{A}$. Consider a path p that is defined as follows. We start with initial state q_0, then go to $q_1 = T(q_0, a)$, then go to $q_2 = T(q_1, a)$ and so on. We stop this procedure when the same state occurs twice in p. Let $p = q_0, \ldots, q_k, q_{k+1}, \ldots, q_{m-1}, q_m$ where $q_m = q_k$.

Now for each $i \leqslant m - 1$ we will define a language L_i which is either ultimately periodic or empty. If $q_i \notin F$ then $L_i = \varnothing$. If $q_i \in F$ and $i < k$ then $L_i = \{a^i\}$. If $q_i \in F$ and $k \leqslant i \leqslant m-1$ then $L_i = \{a^{i+n(m-k)} \mid n \in \omega\}$. It is not hard to see now that $L = \bigcup_{i \leqslant m-1} L_i$. Therefore, L is a finite union of ultimately periodic languages.

Question 7.

(a) $r = (a+b)^* \cdot a \cdot (a+b) \cdot (a+b) \cdot (a+b) \cdot (a+b) \cdot (a+b) \cdot (a+b) \cdot b \cdot (a+b)^*$.

(b) $r = a^* \cdot b^* \cdot a^* \cdot a$.

(c) $r = (a+b)^* \cdot a \cdot (a+b) \cdot (a+b) \cdot (a+b) \cdot (a+b) \cdot (a+b) \cdot (a+b) \cdot (a+b) \cdot (a+b)$.

Question 8.

(a) **(b)**