Big-Oh notation: few examples

Example 1: Prove that running time $T(n) = n^3 + 20n + 1$ is $O(n^3)$

Proof: by the Big-Oh definition, $T(n)$ is $O(n^3)$ if $T(n) \le c \cdot n^3$ for some $n \ge n_0$. Let us check this condition: if $n^3 + 20n + 1 \le c \cdot n^3$ then $1 + \frac{20}{n^2} + \frac{1}{n^3} \le c$. Therefore, the Big-Oh condition holds for $n \ge n_0 = 1$ and $c \ge 22$ (= 1 + 20 + 1). Larger values of n_0 result in smaller factors c (e.g., for $n_0 = 10$ $c \ge 1.201$ and so on) but in any case the above statement is valid.

Example 2: Prove that running time $T(n) = n^3 + 20n + 1$ is not $O(n^2)$

Proof: by the Big-Oh definition, $T(n)$ is $O(n^2)$ if $T(n) \le c \cdot n^2$ for some $n \ge n_0$. Let us check this condition: if $n^3 + 20n + 1 \le c \cdot n^2$ then $n + \frac{20}{n} + \frac{1}{n^2} \le c$. Therefore, the Big-Oh condition cannot hold (the left side of the latter inequality is growing infinitely, so that there is no such constant factor c).

Example 3: Prove that running time $T(n) = n^3 + 20n + 1$ is $O(n^4)$

Proof: by the Big-Oh definition, $T(n)$ is $O(n^4)$ if $T(n) \le c \cdot n^4$ for some $n \ge n_0$. Let us check this condition: if $n^3 + 20n + 1 \le c \cdot n^4$ then $\frac{1}{n} + \frac{20}{n^3} + \frac{1}{n^4} \le c$. Therefore, the Big-Oh condition holds for $n \ge n_0 = 1$ and $c \ge 22$ (= 1 + 20 + 1). Larger values of n_0 result in smaller factors c (e.g., for $n_0 = 10$ $c \ge 0.10201$ and so on) but in any case the above statement is valid.

Example 4: Prove that running time $T(n) = n^3 + 20n$ is $\Omega(n^2)$

Proof: by the Big-Omega definition, $T(n)$ is $\Omega(n^2)$ if $T(n) \ge c \cdot n^2$ for some $n \ge n_0$. Let us check this condition: if $n^3 + 20n \ge c \cdot n^2$ then $n + \frac{20}{n} \ge c$. The left side of this inequality has the minimum value of 8.94 for $n = \sqrt{20} \cong 4.47$ Therefore, the Big-Omega condition holds for $n \ge n_0 = 5$ and $c \le 9$. Larger values of n_0 result in larger factors c (e.g., for $n_0 = 10$ $c \le 12.01$) but in any case the above statement is valid.