Typical Complexity Curves

$T(n) \propto \log n$ logarithmic

$T(n) \propto n$ linear

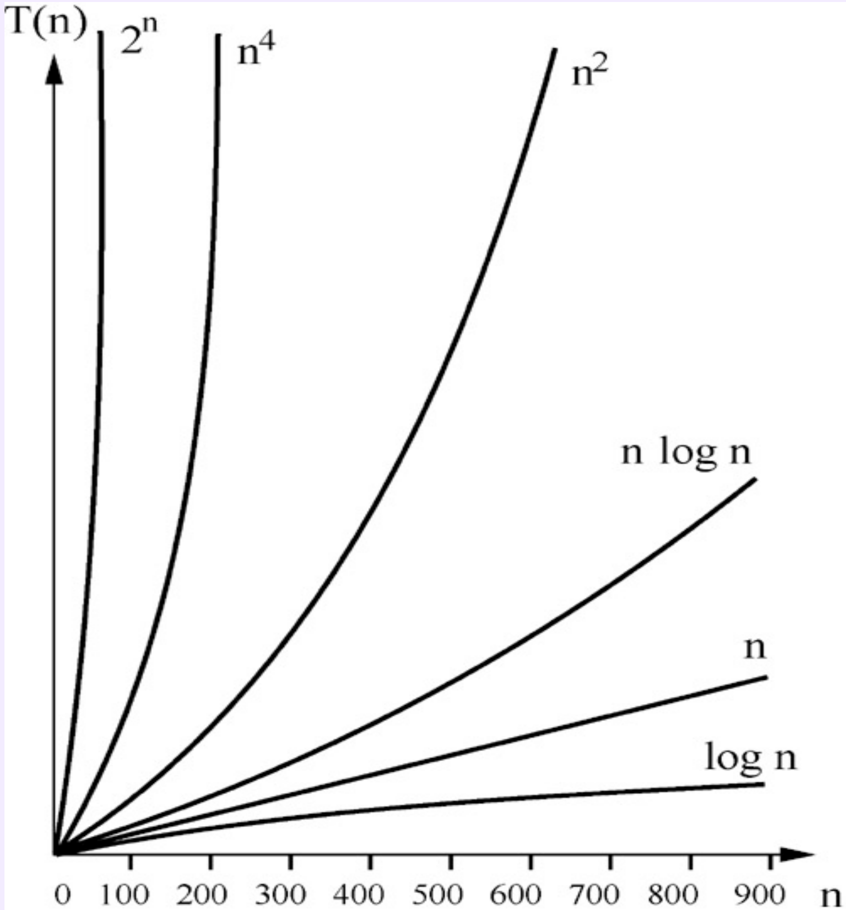
$T(n) \propto n \log n$

$T(n) \propto n^2$ quadratic

$T(n) \propto n^3$ cubic

$T(n) \propto n^k$ polynomial

$T(n) \propto 2^n$ exponential



Relative growth: $g(n) = f(n) / f(5)$

Function $f(n)$		Input size n: 5	25	125	625
Constant	1	1	1	1	1
Logarithm	$\log_5 n$	1	2	3	4
Linear	n	**1**	**5**	**25**	**125**
"n log n"	$n \log_5 n$	1	10	75	500
Quadratic	n^2	1	25 (5^2)	625 (5^4)	15,625 (5^6)
Cubic	n^3	1	125 (5^3)	15,625 (5^6)	1,953,125 (5^9)
Exponential	2^n	**1**	$\mathbf{2^{20} \approx 10^6}$	$\mathbf{2^{120} \approx 10^{36}}$	$\mathbf{2^{620} \approx 10^{187}}$

"Big-Oh" $O(\ldots)$: Formal Definition

Let $f(n)$ and $g(n)$ be nonnegative-valued functions defined on nonnegative integers n

The function $g(n)$ is $O(f)$ (read: $g(n)$ is **Big Oh** of $f(n)$) **iff** there exists a positive real constant c and a positive integer n_0 such that $g(n) \leq cf(n)$ for all $n > n_0$

- Notation "***iff***" is an abbreviation of "**if and only if**"
- *Example* 1.9 (p.15): $g(n) = 100\log_{10}n$ is $O(n)$

  $\Leftarrow g(n) < n$ if $n > 238$ or $g(n) < 0.3n$ if $n > 1000$

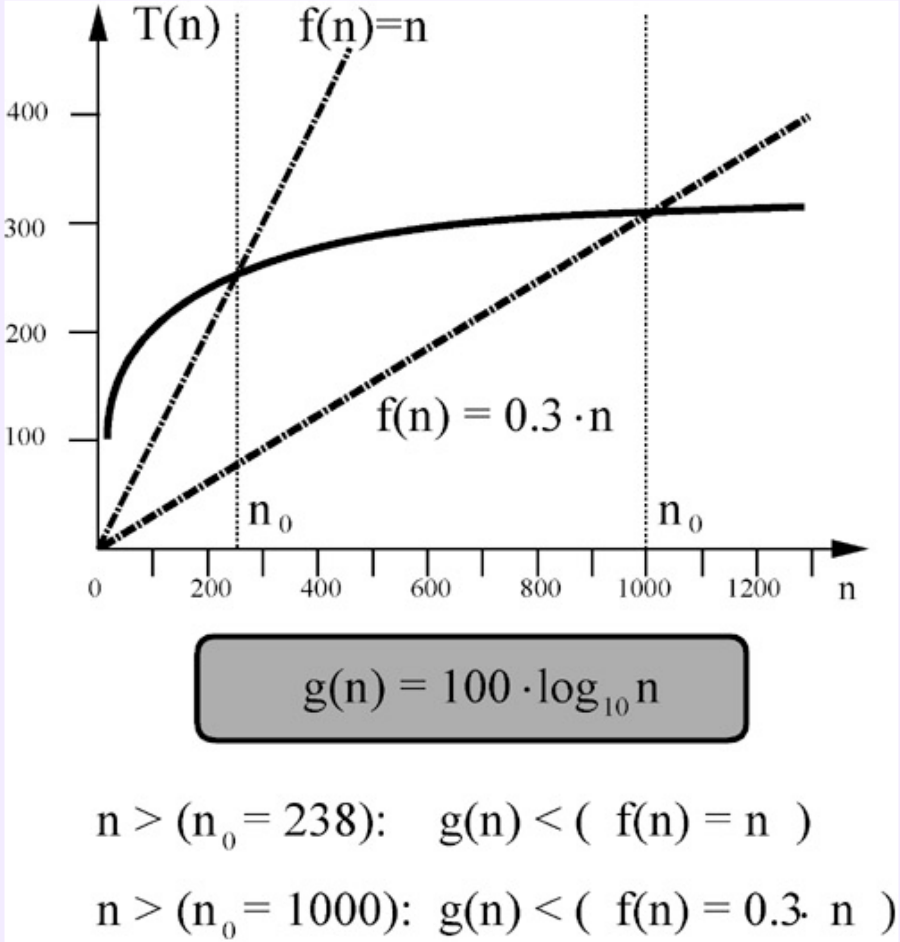
$g(n)$ is $O(f(n))$, or $g(n) = O(f(n))$

$g(n)$ is $O(f(n))$ if:

- a constant $c > 0$ exists such that $cf(n)$ grows faster than $g(n)$ for all $n > n_0$

To prove that some function $g(n)$ is $O(f(n))$ means to show for g and f such constants c and n_0 exist

The constants c and n_0 are interdependent



"Big-Oh" $O(\ldots)$: Informal Meaning

- If $g(n)$ is $O(f(n))$, an algorithm with running time $g(n)$ runs **asymptotically** (i.e. for large n), **at most** as fast, to within a constant factor, as an algorithm with running time $f(n)$

  $O(f(n))$ specifies an <u>asymptotic upper bound</u>, i.e. $g(n)$ for large n may approach closer and closer to $cf(n)$

  Notation $g(n) = O(f(n))$ means actually $g(n) \in O(f(n))$, i.e. $g(n)$ is a member of the set $O(f(n))$ of functions increasing with the same or lesser rate if $n \rightarrow \infty$

Big Omega Ω(...)

- The function $g(n)$ is $\Omega(f(n))$ **iff** there exists a positive real constant c and a positive integer n_0 such that $g(n) \geq cf(n)$ for all $n > n_0$
  - $\Omega(\ldots)$ is opposite to $O(\ldots)$ and specifies an asymptotic lower bound: if $g(n)$ is $\Omega(f(n))$ then $f(n)$ is $O(g(n))$
  - **Example 1**: $5n^2$ is $\Omega(n) \Leftarrow 5n^2 \geq 5n$ for $n \geq 1$
  - **Example 2**: $0.01n$ is $\Omega(\log n) \Leftarrow 0.01n \geq 0.5\log_{10}n$ for $n \geq 100$

Big Theta Θ(...)

- The function $g(n)$ is $\Theta(f(n))$ **iff** there exists two positive real constants c_1 and c_2 and a positive integer n_0 such that $c_1 f(n) \leq g(n) \leq c_2 f(n)$ for all $n > n_0$

$$g(n) \text{ is } \Theta(f(n)) \Rightarrow$$

$$g(n) \text{ is } O(f(n)) \text{ AND } f(n) \text{ is } O(g(n))$$

Ex.: the same rate of increase for $g(n) = n + 5n^{0.5}$ and $f(n) = n$

$$\Rightarrow \quad n \leq n + 5n^{0.5} \leq 6n \text{ for } n > 1$$

Comparisons: Two Crucial Ideas

- Exact running time function is unimportant since it can be multiplied by an arbitrary positive constant.
- Two functions are compared ***asymptotically***, for large n, and not near the origin
  - If the constants c involved are very large, then the asymptotical behaviour is of no practical interest!
  - To prove that $g(n)$ is **not** $O(f(n))$, $\Omega(f(n))$, or $\Theta(f(n))$ we have to show that the desired constants do not exist, i.e. lead to a contradiction

Example 1.12, p.17

Linear function $g(n) = an + b$; $a > 0$, is $O(n)$

To prove, we form a chain of inequalities:

$g(n) \leq a\,n + |b| \leq g(n) \leq (a + |b|) \cdot n$ for all $n \geq 1$

Do not write $O(2n)$ **or** $O(an + b)$ **as this means still** $O(n)$**!**

$O(n)$ - running time:

$T(n) = 3n + 1$ $\quad T(n) = 10^8 + n$

$T(n) = 50 + 10^{-8}\,n$ $\quad T(n) = 10^6\,n + 1$

Remember that "Big-Oh" describes an "asymptotic behaviour" for large problem sizes

Example 1.13, p.17

Polynomial $P_k(n) = a_k n^k + a_{k-1} n^{k-1} + \ldots + a_1 n + a_0$; $a_k > 0$,
is $O(n^k) \Leftarrow P_k(n) \leq (a_k + |a_{k-1}| + \ldots + |a_0|)\, n^k$; $n \geq 1$

Do not write $O(P_k(n))$ **as this means still** $O(n^k)$**!**

$O(n^k)$ - running time:

- $T(n) = 3n^2 + 5n + 1$ is $O(n^2)$ Is it also $O(n^3)$?
- $T(n) = 10^{-8} n^3 + 10^8 n^2 + 30$ is $O(n^3)$
- $T(n) = 10^{-8} n^8 + 1000n + 1$ is $O(n^8)$

$$T(n) = P_k(n) \Rightarrow O(n^m),\ m \geq k;\ \ \Theta(n^k);\ \ \Omega(n^m);\ m \leq k$$

Example 1.14, p.17

Exponential $g(n) = 2^{n+k}$ is $O(2^n)$: $2^{n+k} = 2^k \cdot 2^n$ for all n

Exponential $g(n) = m^{n+k}$ is $O(l^n)$, $l \geq m > 1$:

$$m^{n+k} \leq l^{n+k} = l^k \cdot l^n \text{ for all } n, k$$

A "brute-force" search for the best combination of n interdependent binary decisions by exhausting all the 2^n possible combinations has exponential time complexity! Therefore, **<u>try to find a more efficient way of solving the decision problem</u>** with $n \geq 20 \ldots 30$

Example 1.15, p.17

- Logarithmic function $g(n) = \log_m n$ has the same rate of increase as $\log_2 n$ because

$$\log_m n = \log_m 2 \cdot \log_2 n \quad \text{for all } n, m > 0$$

Do not write $O(\log_m n)$ **as this means still** $O(\log n)$**!**

You will find later that the most efficient search for data in an ordered array has logarithmic time complexity