Min Cut / Max Flow Energy Minimisation

COMPSCI 773 S1 T

VISION GUIDED CONTROL

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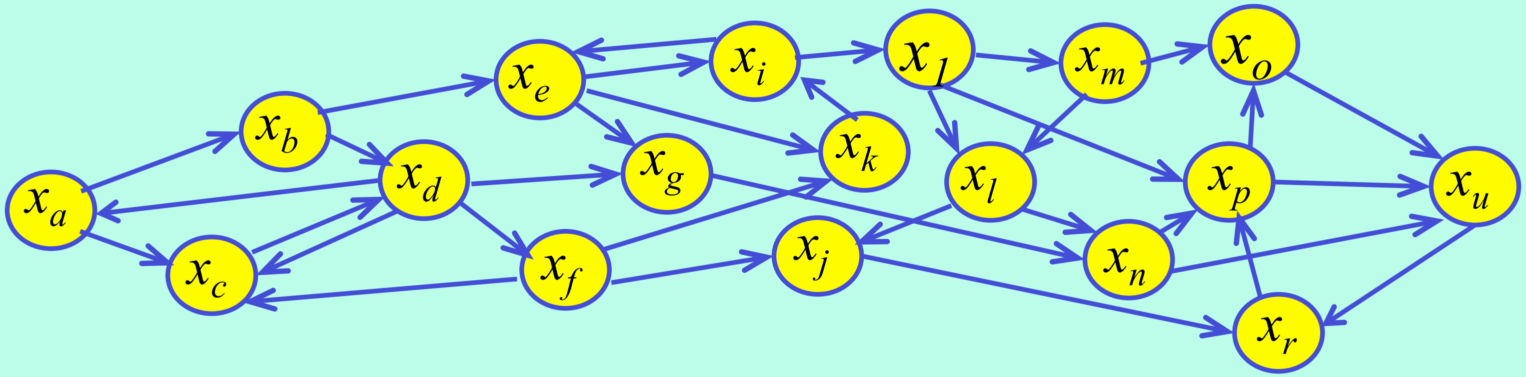
Dissimilarity Minimisation

- 3-D surface by minimising energy (dissimilarity) of stereo images:
  - Combinatorial optimisation on graphs specifying relationships between neighbouring pairs of disparities and image signals
  - Generally, an **NP-hard** problem (the **exponential** complexity)
    - Energy (dissimilarity) accumulates weights of nodes and edges
  - Approximate iterative **polynomial-time** solution
    - **Maximum flow / minimum cut** algorithms applied to special graphs
    - Solution is provably within a fixed factor of the global minimum
    - General **maximum flow** problem for a network, or a directed graph (digraph) **G** with two special nodes: a source, s, and a sink, t

Basic Notation

- **G**=[**N**,**E**] - a digraph (network) with sets of nodes **N** and edges **E**:

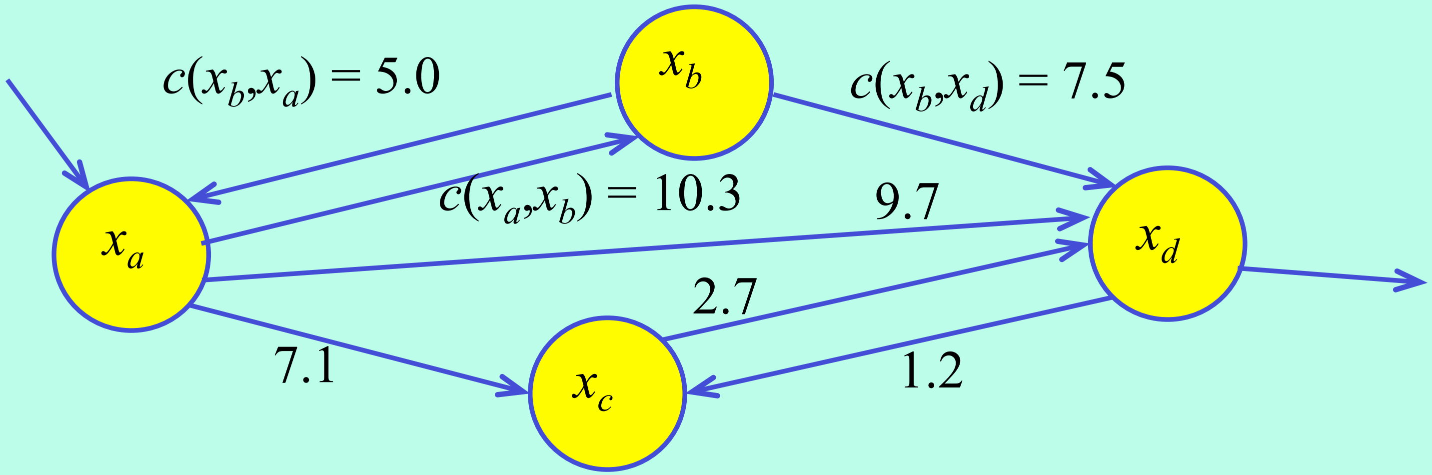


$\mathbf{N} = \{x_a, x_b, x_c, \ldots\}$; $\mathbf{E} = \{(x_a, x_b),(x_a, x_c), \ldots\} \subseteq \mathbf{N}^2$

- ***Chain***: a sequence $x_1,\ldots,x_n$ such that $(x_i, x_{i+1}) \in \mathbf{E}$
- ***Path***: a sequence $x_1,\ldots,x_n$ such that either $(x_i, x_{i+1}) \in \mathbf{E}$ or $(x_{i+1}, x_i) \in \mathbf{E}$
- Set of the subsequent nodes "**after** $\boldsymbol{x}$": $A(x) = \{y \in \mathbf{N} \mid (x, y) \in \mathbf{E}\}$
- Set of the preceding nodes "**before** $\boldsymbol{x}$": $B(x) = \{y \in \mathbf{N} \mid (y, x) \in \mathbf{E}\}$

Flows in Networks

- $c(x,y) \geq 0$ – a *non-negative* **capacity** of $(x,y) \in$ **E**
  c: **E** $\rightarrow$ **R**$^{\geq 0} = [0,\infty)$ – a **capacity function** on **E**



- s, t – the two distinguished nodes (*source*, *sink*)
  – Edges then can be considered as “water pipes”…

Flows in Networks

Static flow of value v from s to t in [**N**; **E**] is a function f: **E** $\rightarrow$ **R**$^{\geq 0}$ satisfying linear conditions:

- The flow through every edge does not exceed the edge capacity

$$\forall_{(x,y)\in \mathbf{E}} \quad f(x,y) \leq c(x,y)$$

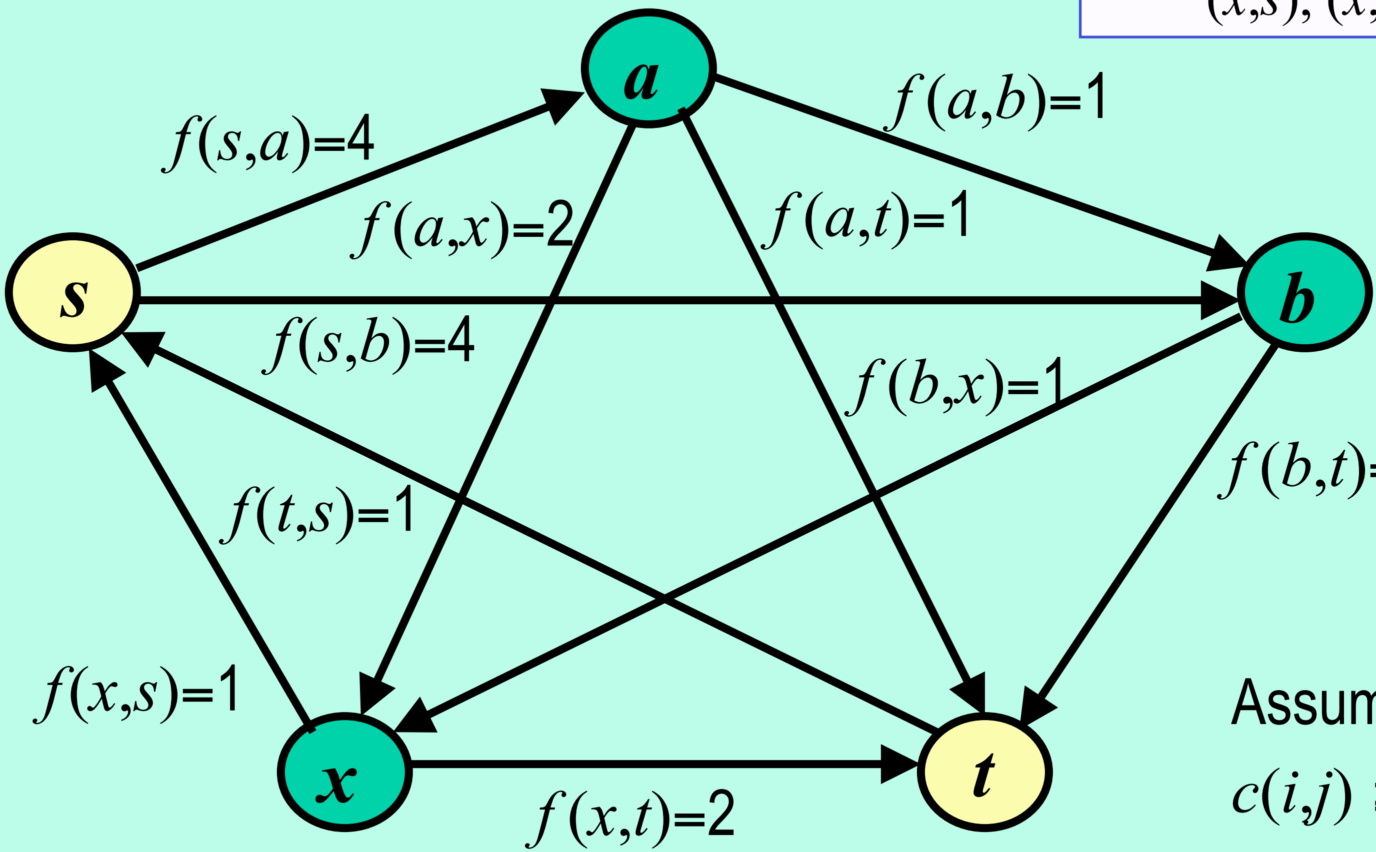
$$\sum_{y\in A(x)} f(x,y) - \sum_{y\in B(x)} f(y,x) = \begin{cases} v & x = s \\ 0 & x \neq s,t \\ -v & x = t \end{cases}$$

- Every node except s and t has equal inflow and outflow

Flow of Value 3

$\mathbf{E} = \{(a,b), (a,t), (a,x),$
$(b,t), (b,x),$
$(s,a), (s,b), (t,s),$
$(x,s), (x,t)\}$

$\mathbf{N} = \{a, b, s, t, x\}$



Assumption:

$c(i,j) \geq f(i,j)$

Static Max Flow Problem

- **Maximise the flow *v* subject to the flow constraints**:

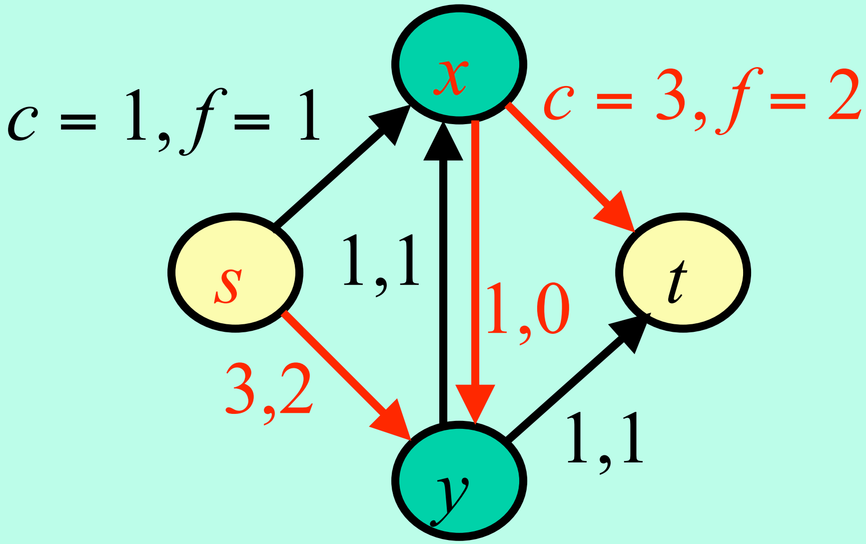
$$\left\{\begin{array}{l} \max v : \forall_{(x,y)\in \mathbf{E}} \quad f(x,y) \le c(x,y) \\ \\ \displaystyle\sum_{y\in A(x)} f(x,y) - \sum_{y\in B(x)} f(y,x) = \begin{cases} v & x = s \\ 0 & x \ne s,t; \quad x \in \mathbf{N} \\ -v & x = t \end{cases} \end{array}\right\}$$

  – A **cut C** of the network [**N**; **E**] is a ***set of edges*** such that their removal separates the source s from the sink t
    - The cut breaks every chain of nodes from the source to the sink

  – The **capacity of the cut C** is the total capacity of its edges, i.e. the sum of their capacities

Cuts and Capacities

Example: the set of edges **C** = $\{(s,y),(x,y),(x,t)\}$ is a cut separating s and t



Capacity of the cut $c(\mathbf{C})$ =
$c(s,y) + c(x,y) + c(x,t) =$
$3 + 1 + 3 = 7$

Flow through the cut $f(\mathbf{C})$ =
$f(s,y) + f(x,y) + f(x,t) =$
$2 + 0 + 2 = 4$

Flow vs. Capacity of the Cut

Lemma 1 [***Ford,Fulkerson***;***1956***]:

Let a flow f from the source s to the sink t in a network [**N**;**E**] have value v

Let **C** be a cut that separates s from t

Then the difference between the forward flow $f_{s\text{-}t}(\mathbf{C})$ from s to t through **C** and the reverse flow $f_{t\text{-}s}(\mathbf{C})$ from t to s through **C** is equal to v and is not greater than the capacity of the cut:

$$v = f_{s\text{-}t}(\mathbf{C}) - f_{t\text{-}s}(\mathbf{C}) \le c(\mathbf{C})$$

Meaning of Lemma 1

The equality in Lemma 1:

the value v of a flow from the source s to the sink t is equal to the **net flow** across any cut separating s and t

The inequality in Lemma 1:

the net flow across any cut separating s and t does not exceed the capacity of the cut

Thus, the net flow from s to t is bounded by the capacities of the cuts separating s and t

Maximal Flow / Minimum Cut

Max-flow min-cut theorem [**Ford,Fulkerson;1956**]: For any network the maximum flow value from s to t is equal to the minimum cut capacity of all cuts separating s and t

Corollary 1: A flow is **maximum** if and only if (**iff**) there is no *flow augmenting path* with respect to f

- A path from s to t is a ***flow augmenting path*** w.r.t. a flow f if $f < c$ on forward edges of the path and $f > 0$ on reverse edges of the path
- **Fundamental importance of the corollary**: to increase the value of a flow, improvements are of a very restricted kind!

Maximal Flow / Minimum Cut

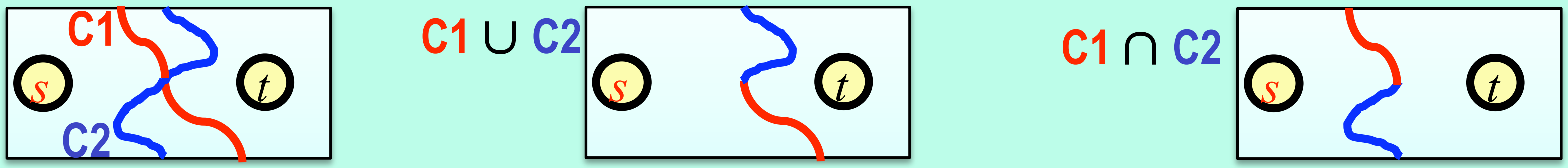
- An edge (x,y) is *saturated* w.r.t. a flow f if $f(x,y) = c(x,y)$
  and is *flowless* w.r.t. f if $f(x,y) = 0$

Corollary 2: A cut **C** is **minimum** iff every maximum flow f saturates all forward edges of the cut whereas all reverse edges of the cut are flowless w.r.t. f

- ***Meaning of Corollary 2*: there are no flow augmenting paths w.r.t. the maximum flow**
- The case of many sources and sinks with unrestricted flows is equivalent to a single source, single sink case

Maximal Flow / Minimum Cut

- Union (∪) of two cuts:
  the set of edges between the union of all the source-side nodes from each cut and all the other nodes in **N**
- Intersection (∩) of two cuts:
  the set of edges between the intersection of the source-side nodes in these cuts and all the other nodes in **N**



Corollary 3: If **C1** and **C2** are minimum cuts, then the union **C1 ∪ C2** and intersection **C1 ∩ C2** are also minimum cuts

Ford–Fulkerson Labelling Algorithm

- Proof of the **max-flow / min-cut theorem** provides, under mild restrictions on the capacity function, a simple efficient algorithm for constructing a maximal flow and minimal cut in a network
- **Initialization**: the zero flow
- **Sequence of "labellings"** (*Routine* A), each of which
  - either results in a flow of higher value (*Routine* B) or
  - terminates with the conclusion that the present flow is maximal (to ensure termination: integer capacities!)

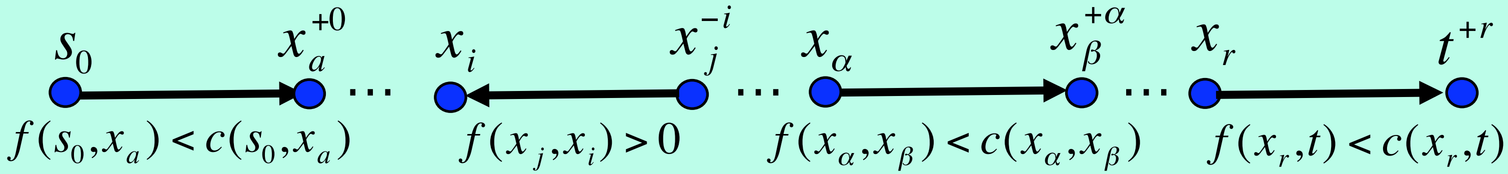
Informal Algorithm Description

- Main idea of labelling (routine A): use a system of labels to find paths between the source and the sink with unsaturated edges
  - Labelling begins from the source (getting the label 0)
  - Let a node x_i be already labelled

1. A subsequent node x_j is not labelled if the edge (x_i, x_j) is saturated; otherwise ($f(x_i, x_j) < c(x_i, x_j)$) it is labelled with $+i$, that is, x_j^{+i}
2. A preceding node x_j is not labelled if the flow $f(x_j, x_i) = 0$; otherwise ($f(x_j, x_i) > 0$) it is labelled with $-i$, that is, x_j^{-i}

- Therefore, the network flow can be increased by increasing flow through edges ending with (+)-nodes and decreasing it through edges ending in (−)-nodes

Informal Algorithm Description



- If the sink is labelled, then there exists **a flow augmenting path** between the source and the sink such that all its nodes are labelled with the indices of their preceding nodes
  - Because such a path contains only unsaturated edges, all the flows via its edges can be changed by a value

$$h = \min_{\substack{(x_q,x_u^{+q})\in\text{path} \\ (x_k^{-m},x_m)\in\text{path}}} \left\{ c(x_q,x_u^{+q}) - f(x_q,x_u^{+q}), f(x_k^{-m},x_m) \right\} > 0$$

  - The flow via an edge is increased by h if the edge is oriented from s to t (from the source to the sink) and decreased by h otherwise

Ford–Fulkerson Labelling Algorithm

- Given an integral flow f, labels are assigned to nodes of the network
  - Nodes can be unlabelled (**UN**), labelled unscanned (**LUN**), and labelled scanned (**LSN**)
  - A label has one of the forms (x^+, ε) or (x^-, ε), where $x \in$ **N** and ε is a positive integer or infinity (∞)

Routine A: Labelling

- Initially all nodes are unlabelled (**UN**) :

  The source node is **LUN** $(-,\ \varepsilon(s) = \infty)$

  Other nodes are **UN**

Ford–Fulkerson Labelling Algorithm

Routine A: Labelling (cont.)

- For every **LUN** x having the label $(z^{\pm}, \varepsilon(x))$:
  - (1) Convert all **UN** y "after x" (i.e. in $A(x)$) such that $f(x,y) < c(x,y)$ into **LUN** with the labels $(\, x^{+}, \varepsilon(y) = \min[\varepsilon(x),\, c(x,y) - f(x,y)\,]\,)$, and
  - (2) Convert all **UN** y "before x" (i.e. in $B(x)$) such that $f(y,x) > 0$ into **LUN** with the labels $(\, x^{-}, \varepsilon(y) = \min[\varepsilon(x),\, f(y,x)\,]\,)$
  - (3) Such x is now **LSN**
- If the sink t is **LUN, go** to **Routine B;** otherwise (t is **UN**) - stop

Ford–Fulkerson Labelling Algorithm

Routine B: **Flow change** (the sink has been labelled $(y^{\pm},\varepsilon(t))$):

- If t is labelled $(y^{+},\varepsilon(t))$, replace $f(y,t)$ with $f(y,t) + \varepsilon(t)$
- If t is labelled $(y^{-},\varepsilon(t))$, replace $f(t,y)$ with $f(t,y) - \varepsilon(t)$
- In either case,

  if node y is labelled $(x^{+},\varepsilon(t))$, replace $f(x,y)$ with $f(x,y) + \varepsilon(t)$

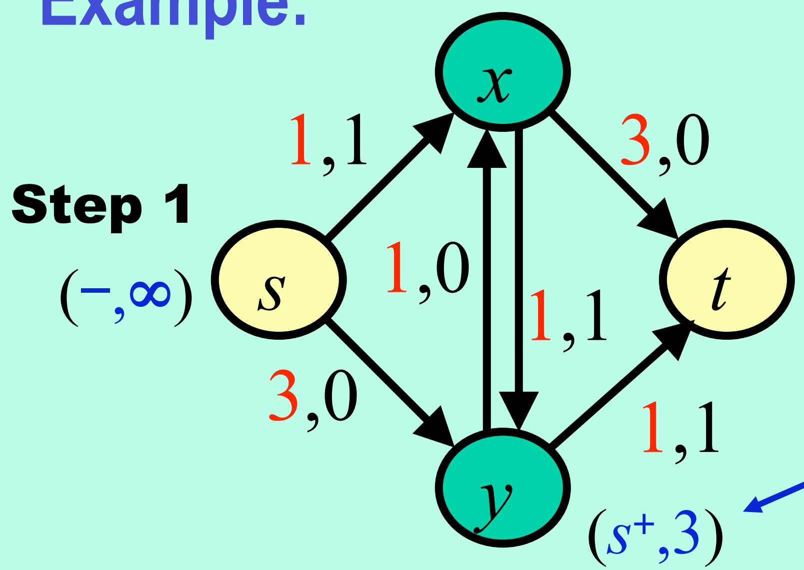
  if node y is labelled $(x^{-},\varepsilon(y))$, replace $f(y,x)$ with $f(y,x) - \varepsilon(t)$ and go on to node x
- Stop the flow change when the source s is reached, discard the old labels, and go back to **Routine A**

Ford–Fulkerson Labelling Algorithm

Labelling searches for a flow augmenting path from s to t :

If **Routine A** ends and the sink is not labelled, the flow is maximum and the set of edges from **UN** to **L*N** nodes is a minimum cut

- **Example:**



Routine A for s:

Node x: $f(s,x) = c(s,x) = 1$

Node y: $f(s,y) = 0 < c(s,y) = 3$

Step 2

$(s^+, \underbrace{\min\{\infty, 3 - 0\}}_{\min\{\varepsilon(s),\, c(s,y) - f(s,y)\}} = 3)$

Ford–Fulkerson Labelling Algorithm

Step 3

Node x: $f(y,x) = 0 < c(y,x) = 1$

$\min\{\varepsilon(y), c(y,x) - f(y,x)\}$

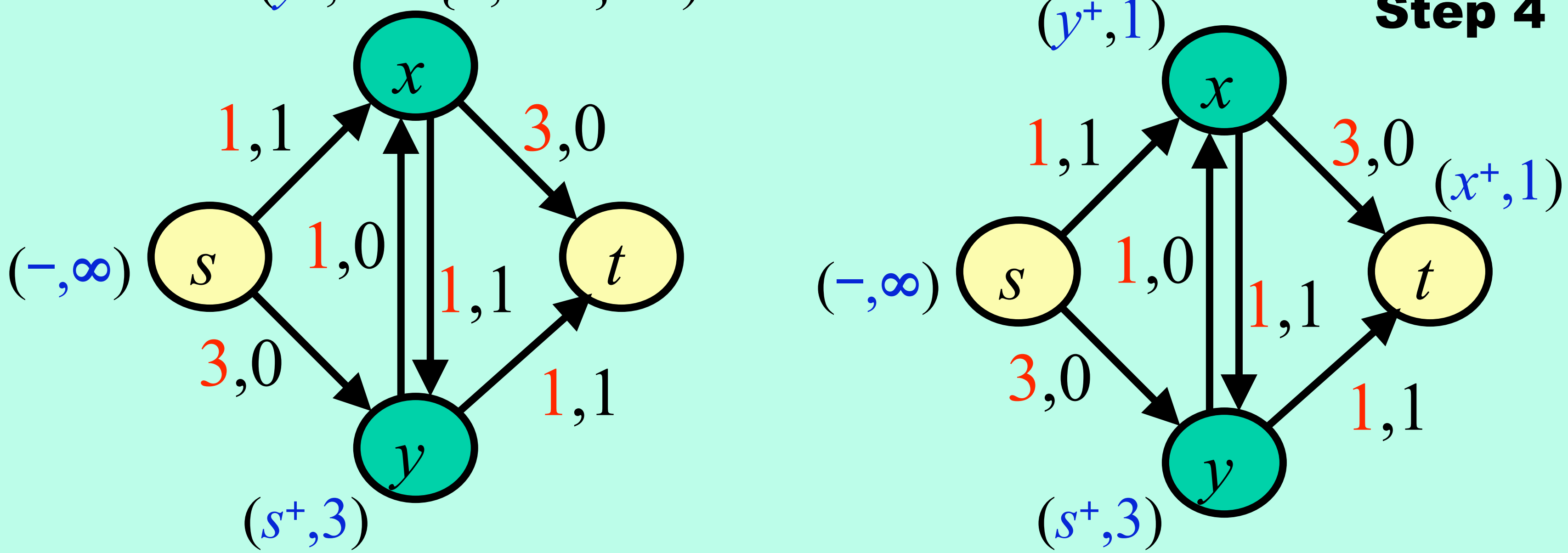
$(y^+, \min\{3, 1-0\} = 1)$

Step 4

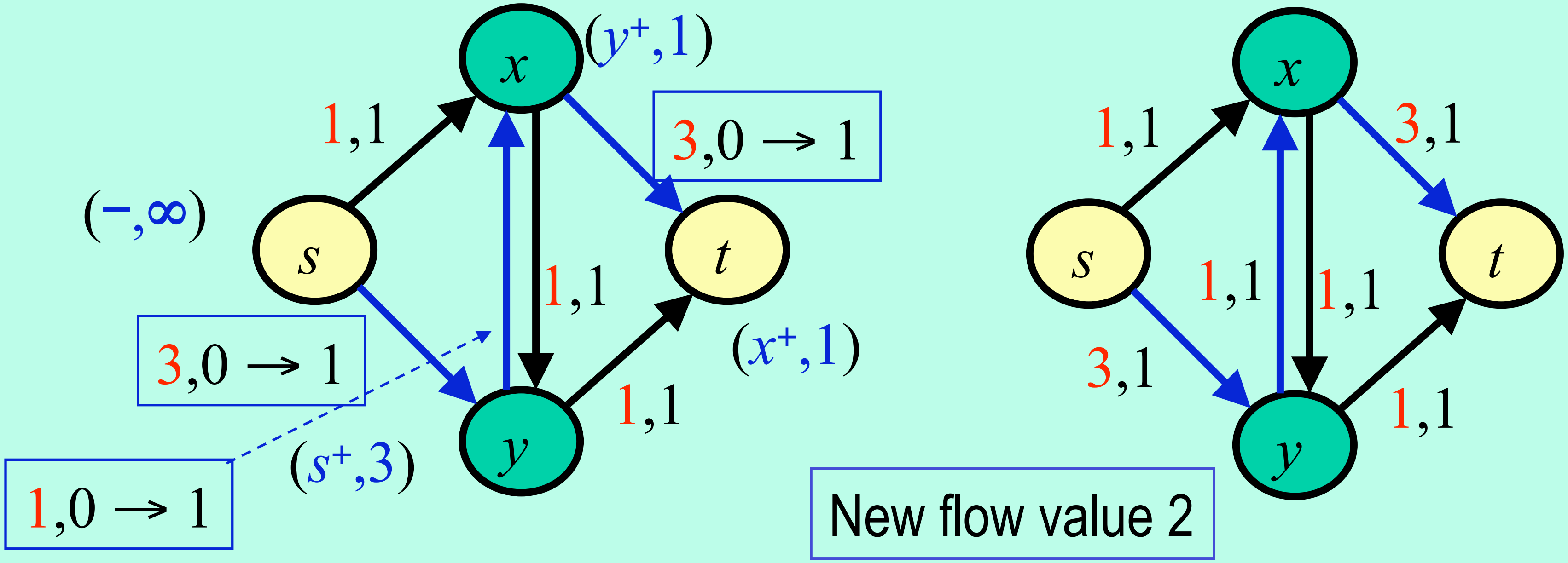
Node x: $f(x,y) = c(x,y) = 1$

Node t: $f(y,t) = c(y,t) = 1$

$f(x,t) = 0 < c(x,t) = 3$

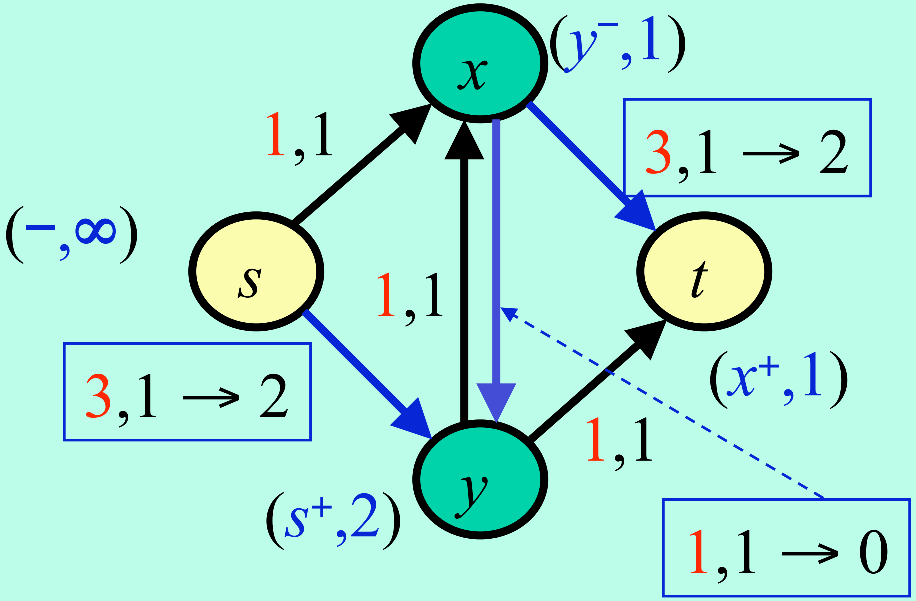


Ford–Fulkerson Labelling Algorithm

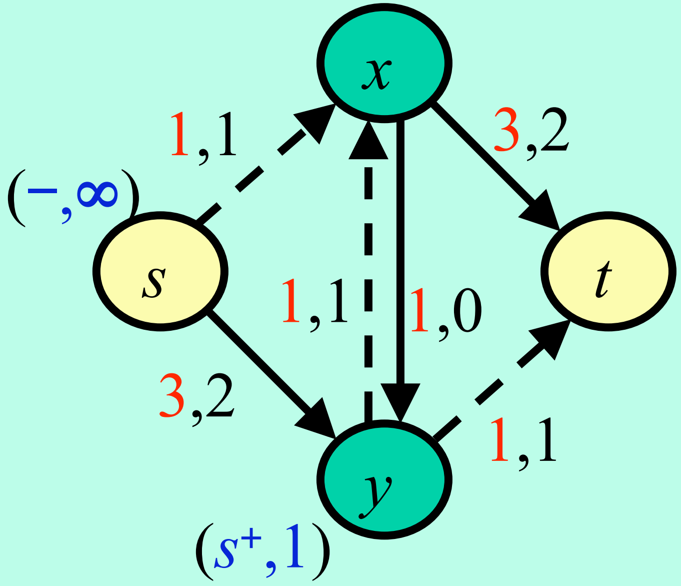


A flow augmenting path is located by backtracking from the sink t according to directions given in labels along which a flow change of $\varepsilon(t) = 1$ can be made

Ford–Fulkerson Labelling Algorithm



New flow value 3



New labelling:
non-sink stop

Polynomial-Time Max-Flow

Maximum flow in the n-node , m-edge network (graph):

- Ford–Fulkerson (finding augmenting paths;1956): $O(nm^2)$
- Dinic (shortest augmenting paths in 1 step; 1970): $O(n^2m)$
  - Graphs: dense – $O(n^3)$; sparse – $O(nm \log n)$
- Goldberg–Tarjan (pushing a pre-flow; 1985): $O\left(nm \log\left(n^2/m\right)\right)$
  - Karzanov's **pre-flow**: the flow in and out of nodes may not be equal (the difference at node j is called **the excess at** j)
  - **Aggressive-passive mode**: push as much as possible into the graph, then trim the excess to 0; no flow until the end

Goldberg–Tarjan Push-Label Algorithm

Relabelling eventually makes 0 the excess at each node

- **Excess** $e(x)$ at x is $e(x) = \sum_{y \in B(x)} f(y,x) - \sum_{y \in A(x)} f(x,y)$
  - The node x is **active** if $e(x) > 0$
  - The source and sink are never active
- **Residual capacity** of an edge (x,y):

  $$r(x,y) = c(x,y) - f(x,y) + f(y,x)$$

  - **Residual network** (graph): **RG** $= \{(x,y) : r(x,y) > 0\}$
- **Distance function** d:**N**→**R** for the nodes:

  (1) $d(t) = 0$;

  (2) if $(x,y) \in$ **E** and $c(x,y) > 0$ then $d(x) \leq d(y) + 1$

Goldberg–Tarjan Push-Label Algorithm

- **Initialisation**:
  - $d(s) = |\mathbf{N}|$ (the number of nodes in a network)
  - $d(t) = 0$
  - $d(x) = 1$ for all $x \neq s,t$
  - $f(s,x) = c(s,x)$ for every edge $(s,x) \in \mathbf{E}$
- **Processing** while an active node ($e(x) > 0$) exists:
  - Select an active node x and
  - Try to push more pre-flow towards the sink

Goldberg–Tarjan Push-Label Algorithm

- **Processing** (cont.):

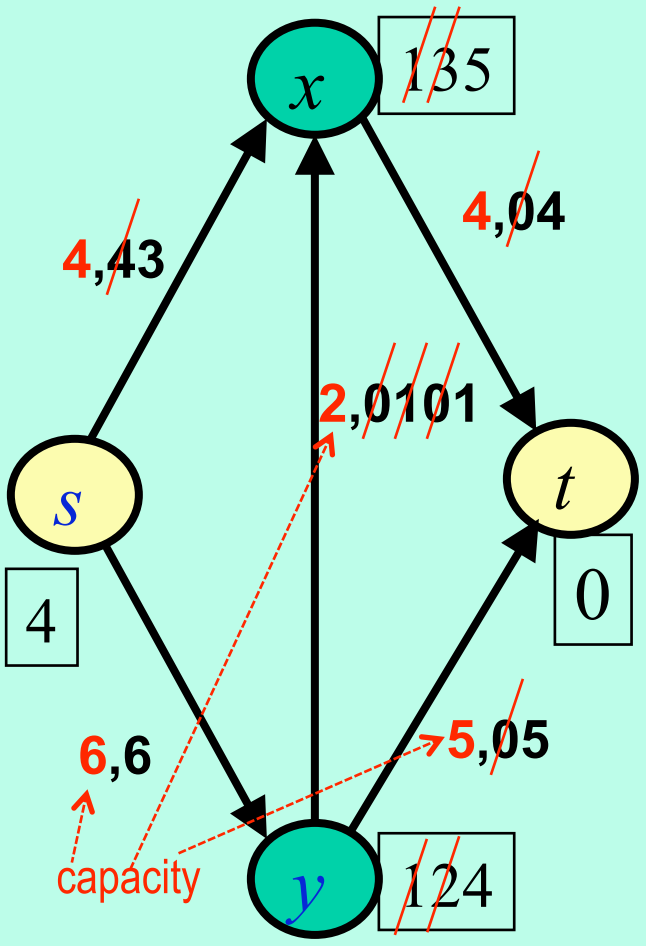
  If $(x,y) \in$ **E**, $d(x) = d(y) + 1$, $r(x,y)$ **>** 0 or
  $(y,x) \in$ **E**, $d(x) = d(y) + 1$, $r(x,y) > 1$, then

  push $\min\{e(x), r(x,y)\}$ from x to y and change f accordingly (pushing as much as the excess at the node and the residual capacity of or from the edge (x,y) allows)

  If nothing can be pushed from x, relabel x by replacing $d(x)$ with $\min\{\ d(y) + 1 : (x,y) \in A(x)$ **and** $r(x,y)$ **>** $0\ \}$

- Once processing is finished, the pre-flow is a **max flow**

Goldberg–Tarjan Push-Label Algorithm



Iter.	*x*	*y*	*s,x*	*s,y*	*y,x*	*x,t*	*y,t*	Active
1	1	1	4	6	0	0	0	*x,y*
2	1	1	4	6	0	4	5	*y*
3	1	2	4	6	0	4	5	*y*
4	1	2	4	6	1	4	5	*x*
5	3	2	4	6	1	4	5	*x*
6	3	2	4	6	0	4	5	*y*
7	3	4	4	6	0	4	5	*y*
8	3	4	4	6	1	4	5	*x*
9	5	4	4	6	1	4	5	*x*
10	5	4	3	6	1	4	5	

D.M.Greig, B.T.Porteous, A.H.Seheult: Exact Maximum A Posteriori Estimation for Binary Images. *Journal of the Royal Statistical Society, Ser. B*, Vol.51 (2), pp.271-279, 1989

Energy Minimization via Graph Cuts

- [D. M. Greig e.a., 1989]: Denoising binary images

  Noisy signals $\mathbf{y}=(y_i: i = 1,\ldots,n)$ are conditionally independent, given a noiseless image $\mathbf{x}=(x_i: i = 1,\ldots,n)$:

$$P(\mathbf{y} \mid \mathbf{x}) = \prod_{i=1}^{n} p(y_i \mid x_i) \equiv \prod_{i=1}^{n} p(y_i \mid 0)^{1-x_i}\, p(y_i \mid 1)^{x_i}$$

$$= \prod_{i=1}^{n} p(y \mid 0) \prod_{i=1}^{n} \left({p(y_i \mid 1)} \big/ {p(y_i \mid 0)} \right)^{x_i}$$

  Prior image model: a 2nd-order Markov random field (MRF):

$$P(\mathbf{x}) \propto \exp\left(\tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} \left[x_i x_j + (1 - x_i)(1 - x_j) \right] \right)$$

  where $\beta_{ii} = 0$; $\beta_{ij} = \beta_{ji} \geq 0$ (neighbours if the strict inequality): i.e. the model of more probable identical neighbours ($x_i = x_j$)

Negative Likelihood as Energy

- Log-likelihood (apart from an additive constant):

$$L(\mathbf{x} \mid \mathbf{y}) \propto \ln P(\mathbf{y} \mid \mathbf{x}) P(\mathbf{x})$$

$$= \sum_{i=1}^{n} \lambda_i x_i + \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \beta_{ij} \left[x_i x_j + (1 - x_i)(1 - x_j) \right]$$

where $\lambda_i = \log\{p(y_i|1)/p(y_i|0)\}$

- Bayesian *maximum a priori* (MAP) image estimate:

$$\mathbf{x}^* = \arg\max_{\mathbf{x}} L(\mathbf{x}|\mathbf{y}) = \arg\min_{\mathbf{x}} [-L(\mathbf{x}|\mathbf{y})]$$

  – Pixel-wise simulated annealing, ICM do not approach the MAP even after hundreds or thousands of iterations…

D.M.Greig, B.T.Porteous, A.H.Seheult: Exact Maximum A Posteriori Estimation for Binary Images.
Journal of the Royal Statistical Society, Ser. B, Vol.51 (2), pp.271-279, 1989

Noiseless image $\mathbf{x}$

MRF prior: $\beta_{ij} = \beta$

Noisy image $\mathbf{y}$

MAP: $\beta_{ij} = 0.3$	Stochastic annealing	ICM Iterated conditional modes	MAP: $\beta_{ij} = 0.7$	Stochastic annealing	ICM Iterated conditional modes

MAP: $\beta_{ij} = 1.1$	Stochastic annealing	ICM Iterated conditional modes

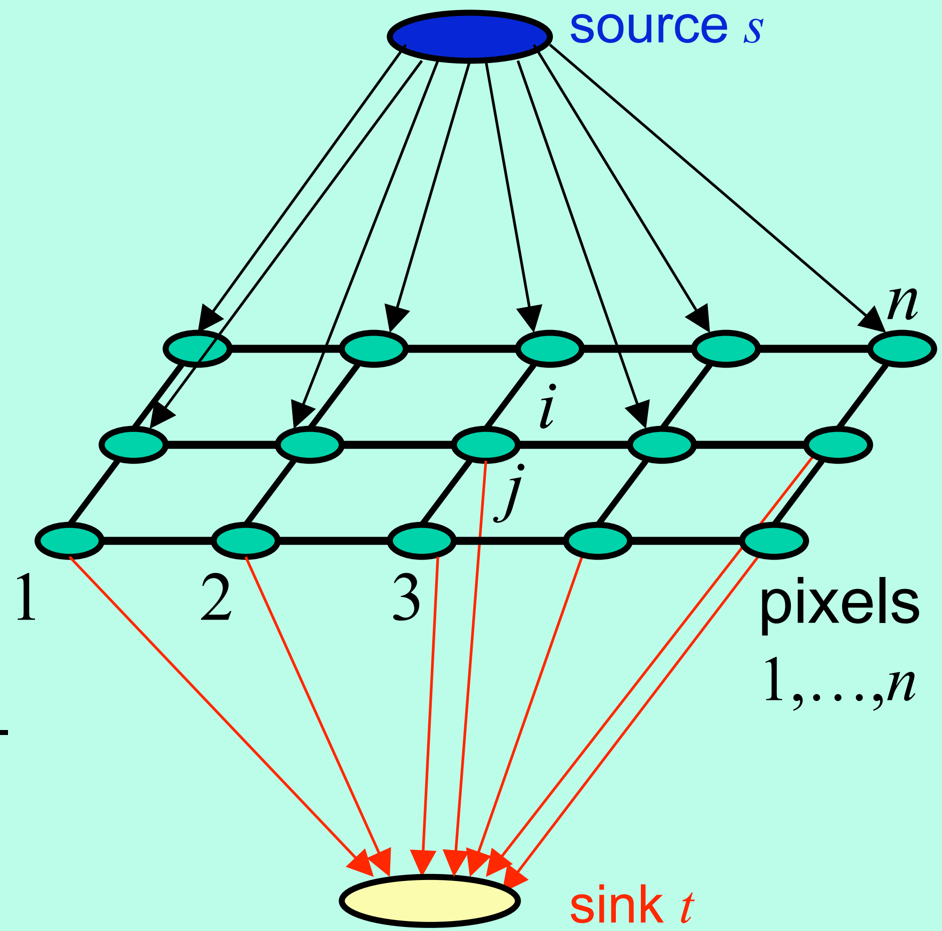
Network Representation

T-links (terminal links):

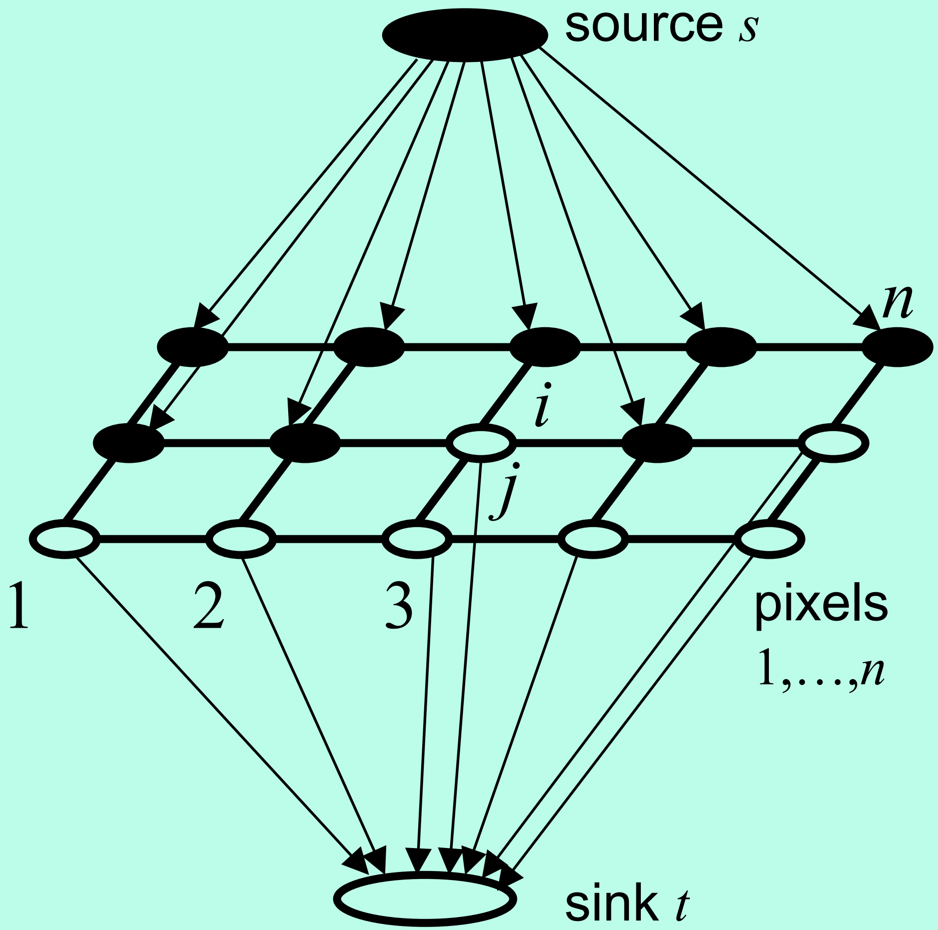
- Directed edge (s,i) with the capacity $c_{si} = \lambda_i$ if $\lambda_i > 0$
- Directed edge (i,t) with the capacity $c_{it} = -\lambda_i$ if $\lambda_i \leq 0$

N-links (neighbouring links):

- Undirected edge (i,j) between two internal nodes – neighbours i and j with the capacity $c_{ij} = \beta_{ij} > 0$



Network Representation



$\lambda_i = \log\{p(y_i|1\}/p(y_i|0)\}$:

$\lambda_i > 0$ for $y_i = 1$ ⬬

and

$\lambda_i < 0$ for $y_i = 0$ ⬭

Example:

$$p(y_i|x_i) = \begin{cases} 0.8 & y_i = x_i \\ 0.2 & y_i \neq x_i \end{cases} \Rightarrow$$

$$\lambda_i = \begin{cases} \log_2 4 = 2 & y_i = 1 \\ -\log_2 4 = -2 & y_i = 0 \end{cases}$$

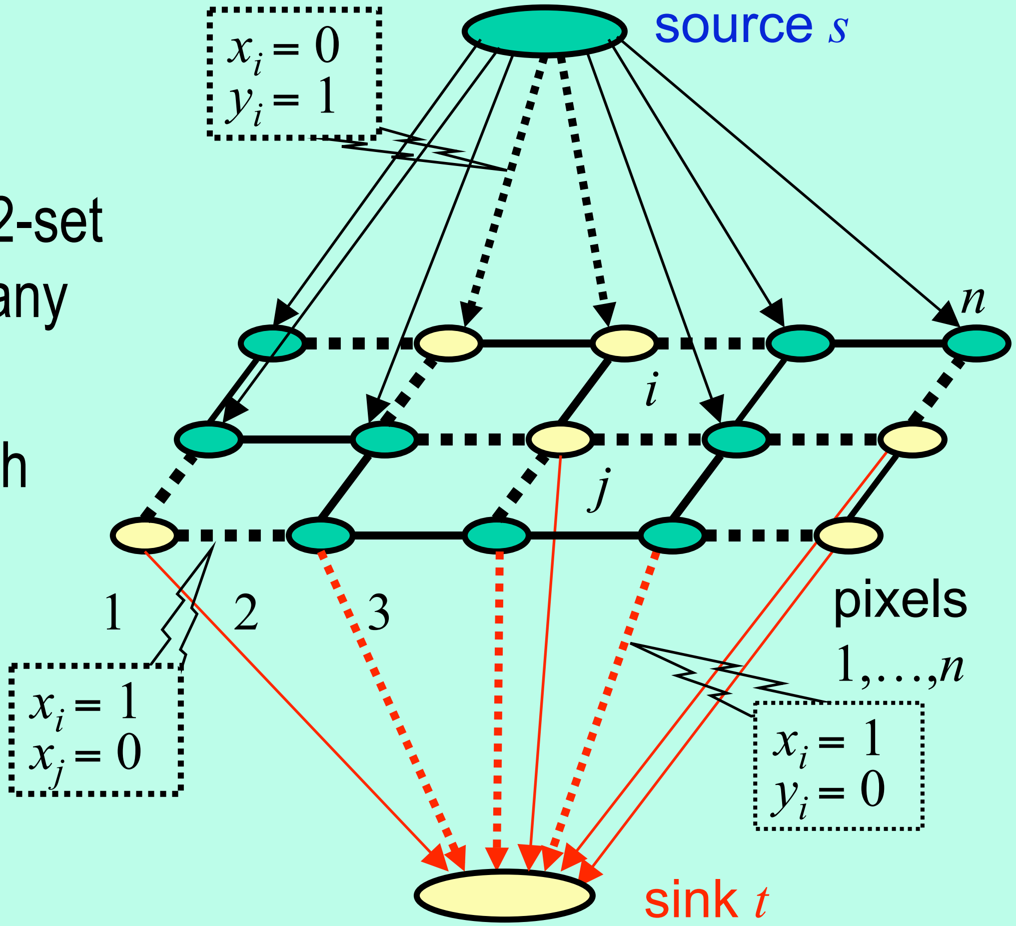
Energy Minimisation via Graph Cut

- $\mathbf{B} = \{s\} \cup \{i: x_i = 1\}$ and
- $\mathbf{W} = \{i: x_i = 0\} \cup \{t\}$ – a 2-set partition of the nodes for any binary image $\mathbf{x}$

Cut - a set of edges (i,j) such that $i \in \mathbf{B}$ and $j \in \mathbf{W}$

Capacity of the cut:

$$C(\mathbf{x}) = \sum_{\substack{(i,j)\in\mathbf{E}:\\ i\in\mathbf{B};\, j\in\mathbf{W}}} c_{ij}$$



Energy Minimisation via Graph Cut

- Capacity of the cut:

$$C(\mathbf{x}) = \sum_{i=1}^{n} \underbrace{x_i \max\{0,-\lambda_i\}} + \sum_{i=1}^{n} \underbrace{(1-x_i)\max\{0,\lambda_i\}} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \underbrace{\beta_{ij}\left(x_i - x_j\right)^2}$$

$x_i = 1$, $y_i = 0$

$x_i = 0$, $y_i = 1$

$x_i = 1$, $x_j = 0$

  - Differs from $[-L(\mathbf{x}|\mathbf{y})]$ by a term that is independent of $\mathbf{x}$
- Maximizing this likelihood is equivalent to minimizing the capacity of the cut, i.e. to finding **the minimum cut**, or **the maximum flow** through the network

Energy Minimisation via Graph Cut

- [Greig e. a., 1989] Accelerated Ford-Fulkerson algorithm:
  - Partitioning the image into $2^K \times 2^K$ connected sub-images
  - Calculate the MAP estimate for each sub-image separately
  - Amalgamate the sub-images to form a set of $2^{K-1} \times 2^{K-1}$ larger sub-images
  - Form the MAP estimate for each of them
  - Continue until the MAP estimate of the complete image
- Simulated annealing: does not necessarily produce a good approximation of an MAP estimate and becomes bogged down by local maxima resulting in under-smooth solutions

Energies Minimised via Graph Cuts

Theorem [Friedman,Drineas,2005; generally, it is a folklore of the combinatorial optimisation: Papadimitriou, Steiglitz, 1986]:

Let $E(x_1,\ldots,x_n) = \Sigma_{i,j}\,\beta_{ij}\,x_i x_j + L$

where $x_i \in \{0,1\}$ and L is linear in x_i plus constants (i.e. $L = \Sigma_i \lambda_i x_i + c$)

Then E can be minimised via graph cut techniques if and only if $\beta_{ij} \leq 0$ for all $i, j \in \{1,2,\ldots, n\}$

Energies Minimised via Graph Cuts

Proof of "if" part ("only if": see [Papadimitriou, Steiglitz, 1986]):

- Energy is rewritten as $E = \Sigma_{i,j=1,\ldots,n}\alpha_{ij}x_i(1-x_j) + \Lambda$
  where $\alpha_{ij} = -\beta_{ij}$ and the linear term Λ is altered
- Minimal E over the binary $x_i \Rightarrow$ a min cut in a complete graph with n nodes and edge weights $w_{ij} = \alpha_{ij}$
  - The cut separates the nodes with $x_i = 0$ from those with $x_j = 1$ (because only $x_i = 1$ and $x_j = 0$ adds α_{ij} to the energy)

Polynomial-time min cut if and only if $w_{ij} \geq 0 \Rightarrow \beta_{ij} \leq 0$

- For the altered linear term $\Lambda = \Sigma_i\gamma_i x_i + \sigma$
  - An edge (s,i) with the weight $w_{si}=\gamma_i$ if $\gamma_i \geq 0$ and an edge (i,t) with $w_{it} = |\gamma_i|$ if $\gamma_i < 0$; therefore, **all weights are non-negative**

Energies Minimised via Graph Cuts

Energy terms depending on signals and pairs of signals (class **F**^{2}):

$$E(x_1,\ldots,x_n) = \Sigma_i E_i(x_i) + \Sigma_{i,j} E_{ij}(x_i, x_j);\ x_i \in \{0,1\}$$

$$E_{ij}(x_i, x_j) \equiv E_{ij}^{00}(1-x_i)(1-x_j) + E_{ij}^{01}(1-x_i)x_j$$
$$+E_{ij}^{10}x_i(1-x_j) + E_{ij}^{11}x_i x_j$$

$$E(x_1,\ldots,x_n) = \sum_{i,j}\left(E_{ij}^{00} + E_{ij}^{11} - E_{ij}^{01} - E_{ij}^{10}\right)x_i x_j + L$$

Such energy function can be minimised via graph cuts if and only if:

$$\underbrace{E_{ij}^{00} + E_{ij}^{11} - E_{ij}^{01} - E_{ij}^{10} \leq 0 \qquad \forall i, j}_{\text{regularity condition}}$$

Large Moves via Min-Cut/Max-Flow

[Boykov,Veksler,Zabih,2001] Approximate energy minimisation by replacing pixel-wise optimising moves with large moves

- Convergence to a solution being provably within a known factor of the global energy minimum
- Energy function minimised w.r.t. a labelling $\mathbf{x} = (x_1,\ldots,x_n)$:

$$E(x_1,\ldots,x_n) = \sum_{i=1}^{n} V_i(x_i) + \sum_{(i,j)\in\mathbf{N}} V_{ij}(x_i,x_j);\ \ x_i \in \mathbf{L};\ \ i = 1,\ldots,n$$

where $\mathbf{L} = \{1, \ldots, L\}$ - an arbitrary finite set of labels

$\mathbf{N} \subset \{1, \ldots, n\}^2$ - a set of neighbouring (interacting) pixel pairs

V_i: $\mathbf{L} \rightarrow \mathbf{R}$ - a pixel-wise potential function (energy of labels)

V_{ij}: $\mathbf{L}^2 \rightarrow \mathbf{R}$ - a pair-wise potential function (energy of label pairs)

Conditions and Optimal Moves

- Arbitrary pixel-wise energies V_i
- **Semimetric** or **metric** pair-wise energies V_{ij}
  - **Semimetric**: $\forall_{\alpha,\beta \in \mathbf{L}}\ V_{ij}(\alpha,\alpha) = 0;\ V_{ij}(\alpha,\beta) = V_{ij}(\beta,\alpha) \geq 0$
  - **Metric**: also the triangle inequality $V_{ij}(\alpha,\beta) \leq V_{ij}(\alpha,\gamma) + V_{ij}(\gamma,\beta)$
- Each labelling $\mathbf{x}$ partitions the pixel set $R = \{1, \ldots, n\}$ into L subsets $R_\lambda = (i \mid i \in R;\ x_i = \lambda \in \mathbf{L}\}$
  - Conditionally optimal large moves change each partition $\boldsymbol{P} = \{R_\lambda : \lambda \in \mathbf{L}\}$ to approach a certain vicinity of the global minimum of the partition energy
    - α–β–swap (with the semimetric V_{ij})
    - α–expansion (with the metric V_{ij})

α,β-swap and α -expansion

α,β **- swap** for an arbitrary pair of labels $\alpha, \beta \in \mathbf{L}$ is a move from a partition $\boldsymbol{P}$ for a current labelling $\mathbf{x}$ to a new partition $\boldsymbol{P'}$ for a new labelling $\mathbf{x}'$ such that $R_\lambda = R'_\lambda$ for any label $\lambda \neq \alpha, \beta$

- Only the labels α and β in their current region $R_{\alpha\beta} = R_\alpha \cup R_\beta$ whereas all other labels in $R \neq R_{\alpha\beta}$ remain fixed.
- In the general case, after an α-β-swap some pixels change their labels from α to β and some others -- from β to α.

α **- expansion** of an arbitrary label α is a move from a partition $\boldsymbol{P}$ for a current labelling $\mathbf{x}$ to a new labelling $\mathbf{x}'$ such that $R_\alpha \subset R'_\alpha$ and $R \backslash R'_\alpha = \cup_{\lambda \in \mathbf{L}; \lambda \neq \alpha} R'_\lambda \subset R \backslash R_\alpha = \cup_{\lambda \in \mathbf{L}; \lambda \neq \alpha} R_\lambda$

- After this move any subset of pixels can change their labels to α

Energy Minimisation Algorithms

Swap algorithm for semimetric interaction potentials

1. **Initialization**: An arbitrary labelling $\mathbf{x}$
2. **Iterative minimization**: For every pair of labels $(\alpha,\beta) \in \mathbf{L}^2$ taken in a fixed or random order:

   2.1 Find $\mathbf{x}^* = \arg\min_{\text{one } \alpha-\beta\text{-swap of } \mathbf{x}} E(\mathbf{x})$ with a min-cut/max-flow technique

   2.2 If $E(\mathbf{x}^*) < E(\mathbf{x})$, then accept the lower-energy labelling: $\mathbf{x} \leftarrow \mathbf{x}^*$
3. **Stopping rule**:

   If a new labelling has been accepted for at least one pair of labels at Step 2.1, continue the minimisation process by returning to Step 2

   Otherwise terminate the process and output the final labelling $\mathbf{x}$

Energy Minimisation Algorithms

Expansion algorithm for metric interaction potentials

1. **Initialization**: An arbitrary labelling $\mathbf{x}$
2. **Iterative minimization**: For every label $\alpha \in$ **L** taken in a fixed or random order:

   2.1 Find $\mathbf{x}^* = \arg\min_{\text{one } \alpha\text{-expansion of } \mathbf{x}} E(\mathbf{x})$ with a min-cut/max-flow technique

   2.2 If $E(\mathbf{x}^*) < E(\mathbf{x})$, then accept the lower-energy labelling: $\mathbf{x} \leftarrow \mathbf{x}^*$
3. **Stopping rule**:

   If a new labelling has been accepted for at least one label at Step 2.1, continue the minimisation process by returning to Step 2

   Otherwise terminate the process and output the final labelling $\mathbf{x}$

$\mathbf{N}_i$ – the set of neighbours of the node i

Optimal Move: Swap Algorithm

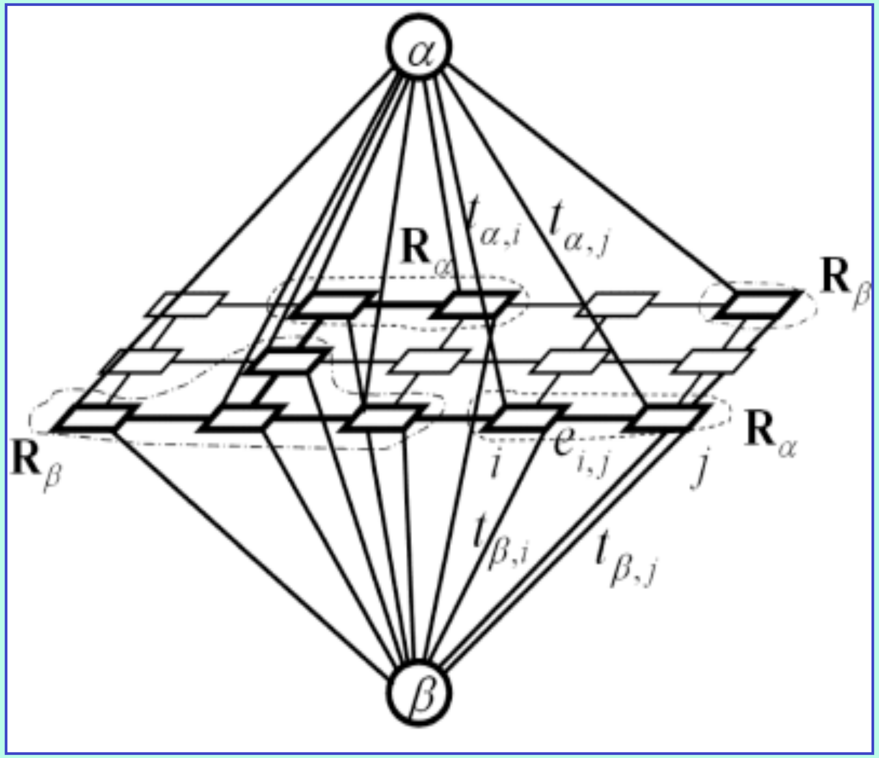
Graph $\mathbf{G}_{\alpha\beta}=[\mathbf{N}_{\alpha\beta};\mathbf{E}_{\alpha\beta}]$ for a set of pixels with the labels α and β

- $\mathbf{N}_{\alpha\beta}$ - two terminals, α and β, and all pixels in $R_{\alpha\beta}$
- Each pixel $i \in R_{\alpha\beta}$ is connected to the terminals by edges (*t*-links) $t_{\alpha,i}$ and $t_{\beta,i}$:

$$\text{weight } t_{\alpha,i} \Leftarrow V_i(\alpha) + \sum_{j\in \mathbf{N}_i;\, j\notin R_{\alpha\beta}} V_{ij}(\alpha, x_j)$$

$$\text{weight } t_{\beta,i} \Leftarrow V_i(\beta) + \sum_{j\in \mathbf{N}_i;\, j\notin R_{\alpha\beta}} V_{ij}(\beta, x_j)$$

- Each neighbour pair $(i,j)\in R_{\alpha\beta}$ is connected by an edge (*n*-link) $e_{i,j}$:



$$\text{weight } e_{i,j} \Leftarrow V_{ij}(\alpha,\beta); (i,j)\in \mathbf{E}_{\alpha\beta} \cap R^2_{\alpha\beta}$$

Optimal Move: Swap Algorithm

A cut **C** on $\mathbf{G}_{\alpha\beta}$ must contain exactly one *t*-link for any pixel $i \in R_{\alpha\beta}$

Otherwise: either there would be a path between the terminals if both the links are included, or

a proper subset of **C** would become a cut if both the links are excluded

Therefore, any cut **C** provides a natural labelling $\mathbf{x}_{\mathbf{C}}$:

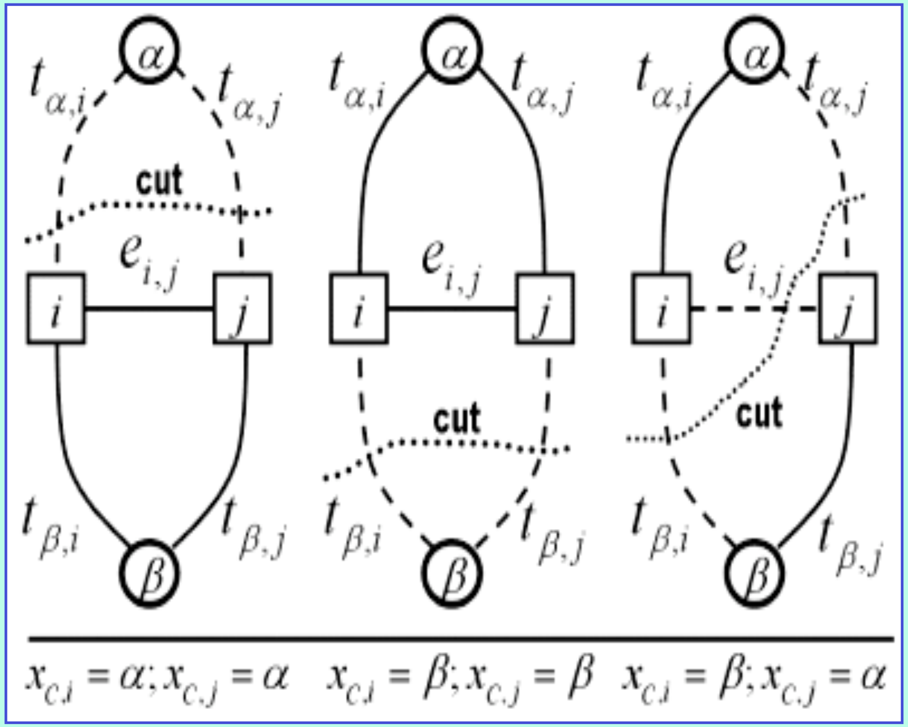
every pixel $i \in R_{\alpha\beta}$ is labelled with α or β if the cut **C** separates i from the terminal α or β, respectively, and the other pixels keep their initial labels:

$$\forall_{i \in R} \quad x_{\mathbf{C},i} = \begin{cases} \alpha & \text{if} \quad i \in R_{\alpha\beta} \quad \text{and} \quad t_{\alpha,i} \in \mathbf{C} \\ \beta & \text{if} \quad i \in R_{\alpha\beta} \quad \text{and} \quad t_{\beta,i} \in \mathbf{C} \\ x_i & \text{if} \quad i \notin R_{\alpha\beta} \end{cases}$$

Optimal Move: Swap Algorithm

Each labelling $\mathbf{x_C}$ corresponding to a cut **C** on $\mathbf{G}_{\alpha\beta}$ is one $\alpha-\beta$-swap from the initial $\mathbf{x}$

- Any *n*-link $e_{i,j}$ is included in a cut **C** only if the pixels i and j are linked to different terminals under the cut



Theorem (BVZ,2001): The capacity $c(\mathbf{C})$ of the cut **C** is the energy function $E(\mathbf{x_C})$ plus a constant

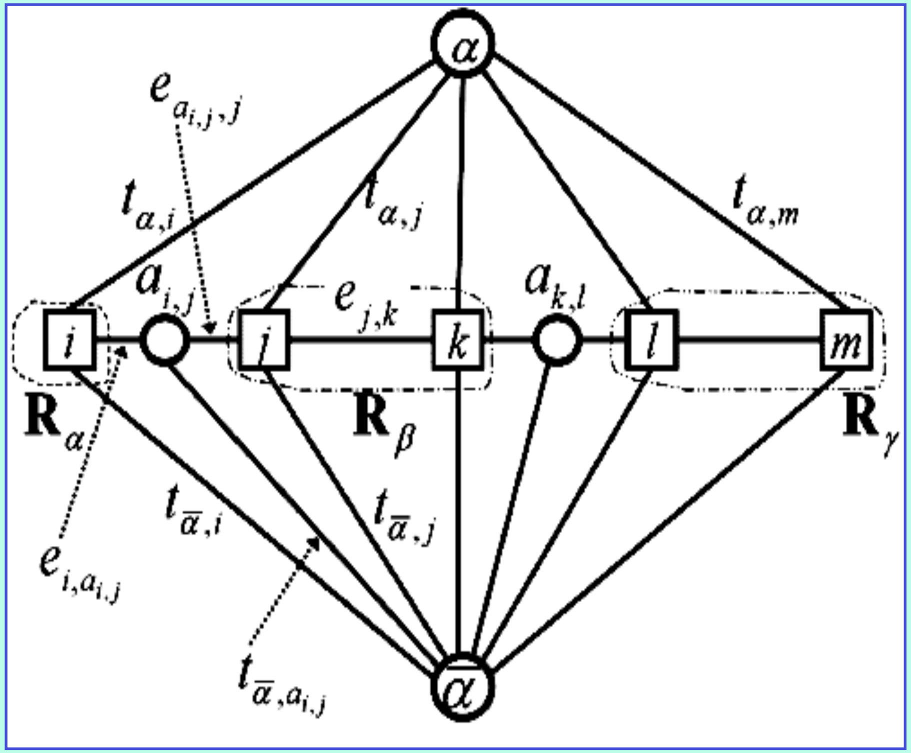
Corollary (BVZ,2001): The lowest energy labelling within a single α-β-swap move from a current labelling $\mathbf{x}$ corresponds to the minimum cut labelling

Optimal Move: Expansion Algorithm

Graph $\mathbf{G}_\alpha=[\mathbf{N}_\alpha;\mathbf{E}_\alpha]$ for a set of pixels with the labels α and $\overline{\alpha}$

- **N**$_\alpha$ - two terminals, α and $\overline{\alpha}$, all pixels $i \in R$, and auxiliary nodes $a_{i,j}$ for each pair (i,j) of the nodes with the labels $x_i \neq x_j$
- Edge weights:

Edge	Weight	Condition
$t_{\bar{\alpha},i}$	∞	$i \in R_\alpha$
$t_{\bar{\alpha},i}$	$V_i(x_i)$	$i \notin R_\alpha$
$t_{\alpha,i}$	$V_i(\alpha)$	$i \in R_\alpha$
$t_{\bar{\alpha},a_{i,j}}$	$V_{ij}(x_i,x_j)$	
$e_{i,a_{i,j}}$	$V_{ij}(x_i,\alpha)$	$(i,j) \in \mathsf{N};\ x_i \neq x_j$
$e_{a_{i,j},j}$	$V_{ij}(\alpha,x_j)$	
$e_{i,j}$	$V_{ij}(x_i,\alpha)$	$(i,j) \in \mathsf{N};\ x_i = x_j$



$R=\{i,j,k,l,m\}$ $R_\alpha=\{i\}$ $R_\beta=\{j,k\}$ $\mathrm{R}_\gamma=\{l,m\}$

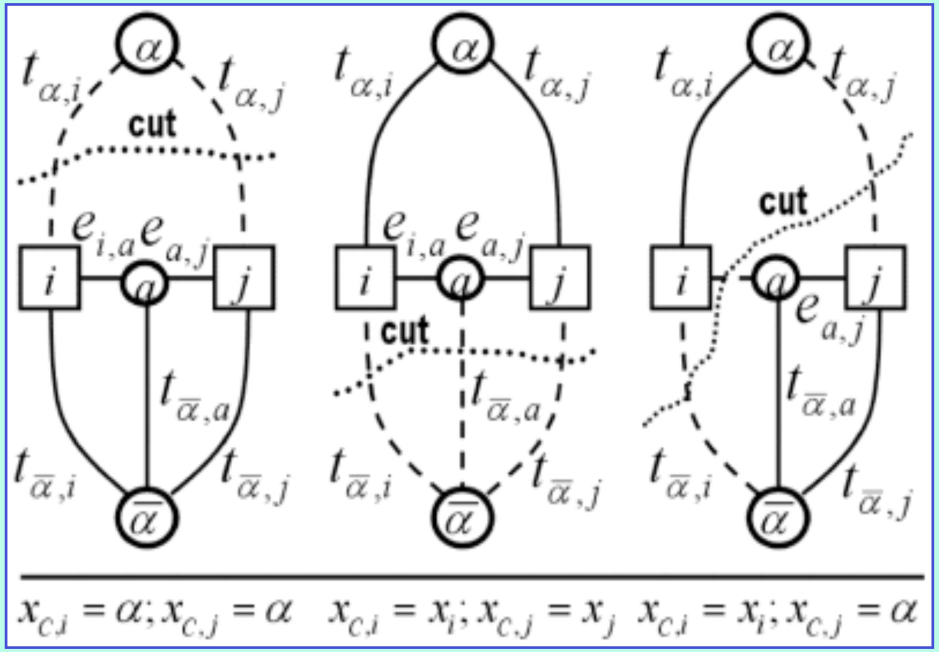
Optimal Move: Expansion Algorithm

Any cut **C** on $\mathbf{G}_\alpha$ must include exactly one *t*-link for any $i \in R$

This provides a natural labelling: $\forall_{i \in R} \; x_{\mathsf{C},i} = \begin{cases} \alpha & \text{if} \quad t_{\alpha,i} \in \mathsf{C} \\ x_i & \text{if} \quad t_{\bar{\alpha},i} \in \mathsf{C} \end{cases}$

- An *n*-link $e_{i,j}$ is in **C** if $i, j \in R$ are connected to different terminals
- The edge triplet $\mathbf{E}_{i,j}$ for $i, j \in R$ such that $x_i \neq x_j$ has the unique minimum cut due to the metric properties of the potentials:

$$
\begin{array}{llll}
\text{if} & t_{\alpha,i}, t_{\alpha,j} \in \mathbf{C} & \text{then} & \mathbf{C} \cap \mathbf{E}_{i,j} = 0 \\
\text{if} & t_{\bar{\alpha},i}, t_{\bar{\alpha},j} \in \mathbf{C} & \text{then} & \mathbf{C} \cap \mathbf{E}_{i,j} = t_{\bar{\alpha},a_{i,j}} \\
\text{if} & t_{\alpha,i}, t_{\alpha,j} \in \mathbf{C} & \text{then} & \mathbf{C} \cap \mathbf{E}_{i,j} = e_{i,a_{i,j}} \\
\text{if} & t_{\alpha,i}, t_{\alpha,j} \in \mathbf{C} & \text{then} & \mathbf{C} \cap \mathbf{E}_{i,j} = e_{a_{i,j},j}
\end{array}
$$



Optimality of Large Moves

- No proven optimality properties for the swap move algorithm
- Local minimum within a fixed factor of the global minimum for the expansion move algorithm
- **Theorem**[Boykov,Veksler,Zabih,2001]:

  Let $\mathbf{x}^*$ and $\mathbf{x}^\circ$ be the labellings for a local energy when the expansion moves are allowed and the global energy minimum, respectively. Then $E(\mathbf{x}^*) \leq 2\gamma E(\mathbf{x}^\circ)$ where

$$\gamma = \max_{(i,j)\in \mathsf{N}} \left(\frac{\max_{\alpha \neq \beta \in \mathsf{L}} V_{i,j}(\alpha, \beta)}{\min_{\alpha \neq \beta \in \mathsf{L}} V_{i,j}(\alpha, \beta)} \right)$$

Pictures from: http://bj.middlebury.edu/~schar/stereo/web/