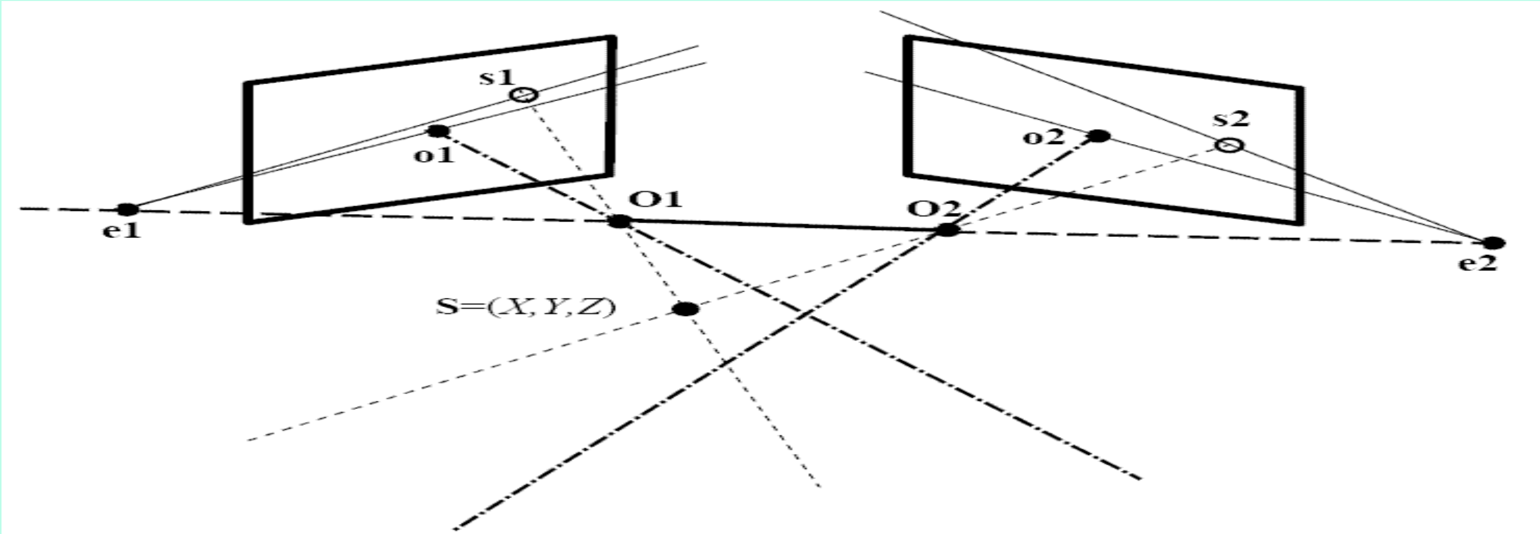
EPIPOLAR GEOMETRY and STEREO VISION

COMPSCI 773 S1T

VISION GUIDED CONTROL

AP Georgy Gimel'farb

Binocular Viewing



$\mathbf{O}_1 = (X_{o,1}, Y_{o,1}, Z_{o,1})$ and $\mathbf{O}_2 = (X_{o,1}, Y_{o,1}, Z_{o,1})$ -optical centres (poles) of cameras

- ***Stereo baseline***: the line segment $\mathbf{O_1O_2}$ between the optical centres
- $\mathbf{o}_1 = (X_{o,1}, Y_{o,1}, Z_{o,1})$ and $\mathbf{o}_2 = (X_{o,2}, Y_{o,2}, Z_{o,2})$ - ***principal points*** of the images
- $\mathbf{e}_1$ and $\mathbf{e}_2$ - ***epipolar points*** in the image planes
  - The projection of one optical centre (pole) onto the plane of another image

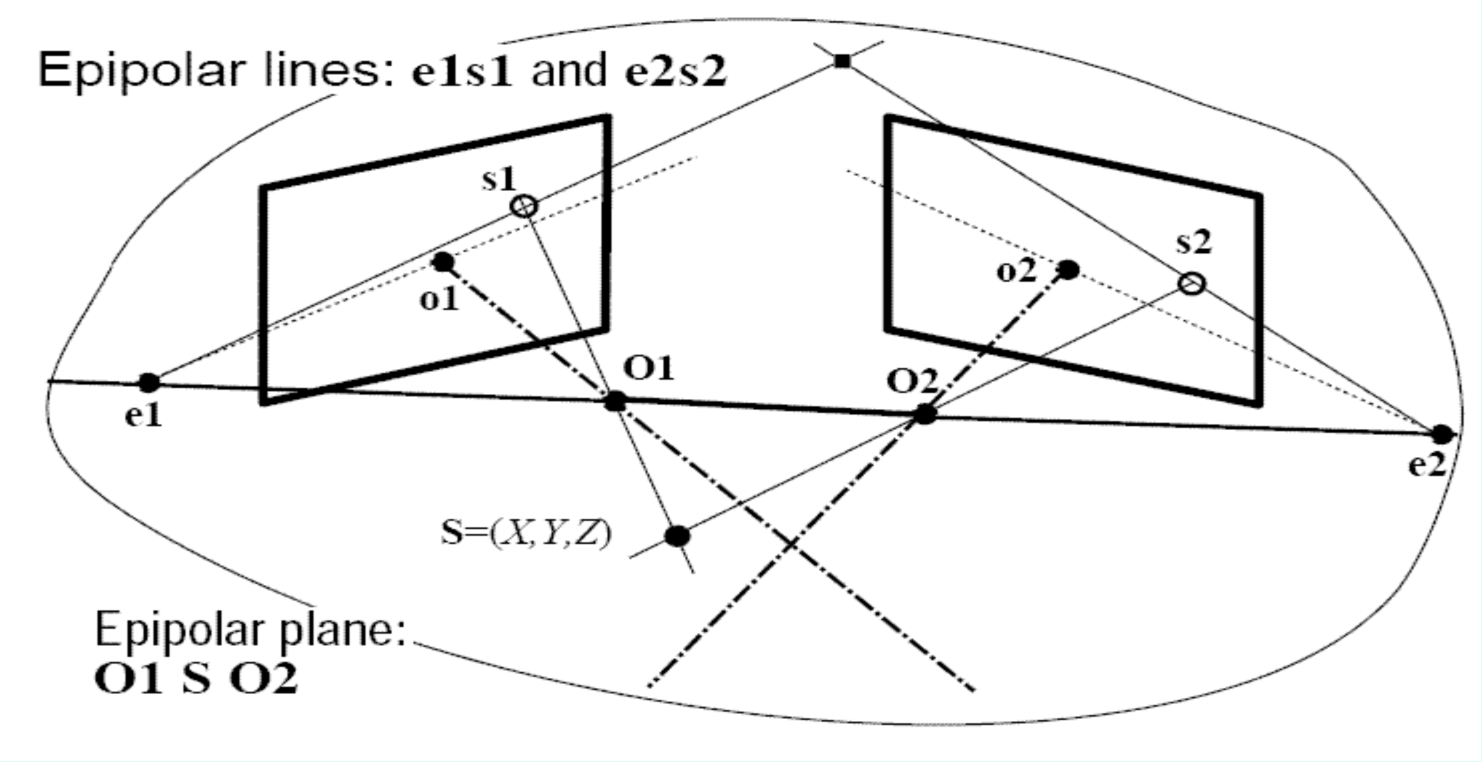
Binocular Viewing

- Conveniently described in terms of corresponding epipolar lines
- *Definition:*
  The ***epipolar line*** through the pixel $\mathbf{s}$ in the image plane is a trace of the intersecting plane containing the 3D point $\mathbf{S}$ and the baseline $\mathbf{O_1O_2}$ (that is, both optical centres $\mathbf{O_1}$ and $\mathbf{O_2}$)
  - Let $\mathbf{s}$ denote the projection of a 3D point $\mathbf{S}$ onto an image plane
  - Any spatial point laying in the plane $\mathbf{SO}_1\mathbf{O}_2$ is projected into the corresponding pair of the epipolar lines in the images
  - For instance, $\mathbf{S}$ is projected into the lines $\mathbf{e_1s_1}$ and $\mathbf{e_2s_2}$

Epipolar Geometry

- *Definition:*
  the ***epipolar profile of the scene*** is the 2D profile of the 3D scene in the intersecting plane $\mathbf{SO_1O_2}$
- Each epipolar profile of the scene is depicted by the corresponding epipolar lines $\mathbf{e_1s_1}$ and $\mathbf{e_2s_2}$ in the images
  - $\mathbf{s_1}$, $\mathbf{s_2}$ - projections of a 3D point $\mathbf{S}$
  - $\mathbf{e_1}$, $\mathbf{e_2}$ - ***epipoles***
    - $\mathbf{e_1}$ – the projection of the optical centre $\mathbf{O_1}$ (or pole) onto the second image
    - $\mathbf{e_2}$ – the projection of the optical centre $\mathbf{O_2}$ (or pole) onto the first image

Epipolar Geometry



- The lines $\mathbf{e_1 s_1}$ and $\mathbf{e_2 s_2}$ are the corresponding epipolar lines

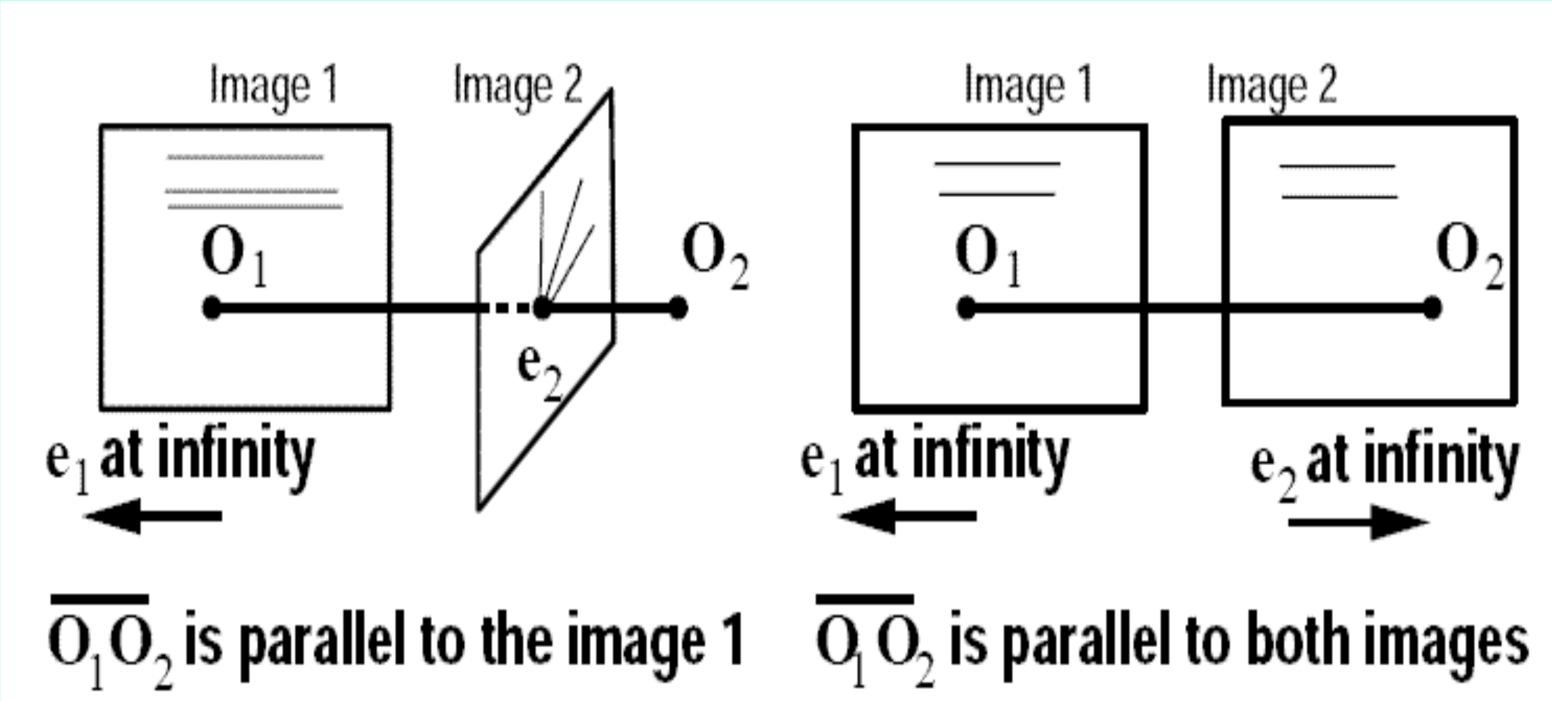
Epipolar Geometry

- **Symmetric epipolar constraint**:
  - For a given point $\mathbf{s}_1$ in the plane of the stereo image 1, all the possible stereo matches in the plane of another image 2 are on the epipolar line passing through the epipole $\mathbf{e}_2$
  - For a given point $\mathbf{s}_2$ in the plane of the stereo image 2, all the possible stereo matches in the plane of another image 1 are on the epipolar line passing through the epipole $\mathbf{e}_1$
  - Corresponding epipolar lines are the intersections of the plane $\mathbf{SO}_1\mathbf{O}_2$ with the image planes

Epipolar Geometry

– *Parallel epipolar lines*:

a special case of a so-called ***horizontal stereo pair***

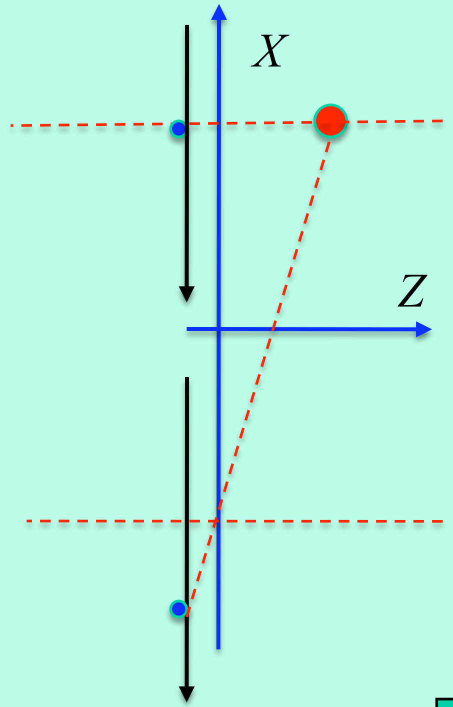


Epipolar Geometry

- Epipolar relations between the image points:
  - Two cameras with the projection matrices $\mathbf{P}_i=[\mathbf{Q}_i\ \mathbf{q}_i]$; $i=1,2$
  - 3D point $\mathbf{S}$ relates to the corresponding image points $\mathbf{s}_1 = \mathbf{P}_1\mathbf{S}$ and $\mathbf{s}_2 = \mathbf{P}_2\mathbf{S}$
- An example:

$$\mathbf{s}_1 = \mathbf{P}_1\mathbf{S} = \begin{bmatrix} 0.5 & 0 & 0 & -1.5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$\mathbf{s}_2 = \mathbf{P}_2\mathbf{S} = \begin{bmatrix} 0.5 & 0 & 0 & 1.5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$$



T indicates the transposition

Epipolar Geometry

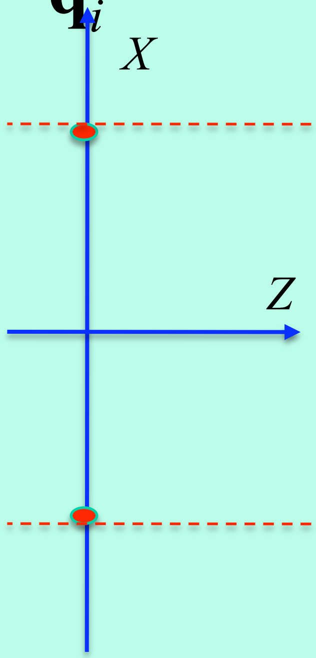
- The optical centre:

$$\mathbf{P}_i[\mathbf{O}_i^{\mathsf{T}}\ 1]^{\mathsf{T}} = \mathbf{Q}_i\mathbf{O}_i + \mathbf{q}_i 1 = \mathbf{0} \rightarrow \mathbf{O}_i = -\mathbf{Q}_i^{-1}\mathbf{q}_i$$

- An example:

$$\mathbf{O}_1 = -\begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -1.5 \\ 0 \\ 0 \end{bmatrix} = -\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1.5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{O}_2 = -\begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1.5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1.5 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix}$$



T indicates the transposition

Epipolar Geometry

- The epipole $\mathbf{e}_j$; $j \neq i$, is given by the relationship:

$$\mathbf{e}_j = \mathbf{P}_j[\mathbf{O}_i^{\mathsf{T}}\ 1]^{\mathsf{T}} = \mathbf{P}_j\ [(-\mathbf{Q}_i^{-1}\mathbf{q}_i)^{\mathsf{T}}\ 1]^{\mathsf{T}}$$

  and is one of the points of each epipolar line:

- An example:

$$\mathbf{e}_1 = \mathbf{P}_1\begin{bmatrix}-3\\0\\0\\1\end{bmatrix} = \begin{bmatrix}0.5 & 0 & 0 & -1.5\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}-3\\0\\0\\1\end{bmatrix} = \begin{bmatrix}-3\\0\\0\end{bmatrix}$$

$$\mathbf{e}_2 = \mathbf{P}_2\begin{bmatrix}3\\0\\0\\1\end{bmatrix} = \begin{bmatrix}0.5 & 0 & 0 & 1.5\\0 & 1 & 0 & 0\\0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}3\\0\\0\\1\end{bmatrix} = \begin{bmatrix}3\\0\\0\end{bmatrix}$$

T indicates the transposition

Epipolar Geometry

- Another point $\mathbf{D}_i$ can be chosen at infinity of the optical ray $\mathbf{O}_i\mathbf{s}_i$, that is, $\mathbf{P}_i[\mathbf{D}_i^{\mathsf{T}}\ \ 0]^{\mathsf{T}} = \mathbf{Q}_i\mathbf{D}_i + \mathbf{q}_i 0 = \mathbf{s}_i \rightarrow \mathbf{D}_i = \mathbf{Q}_i^{-1}\,\mathbf{s}_i$
  - An example:

$$\mathbf{D}_1 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}; \qquad \mathbf{D}_2 = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 2 \end{bmatrix}$$

  - The image $\mathbf{d}_j$ of this point in the second image plane is given by $\mathbf{d}_j = \mathbf{P}_j[\mathbf{D}_i^{\mathsf{T}}\ \ 0]^{\mathsf{T}} = \mathbf{Q}_j\mathbf{Q}_i^{-1}\,\mathbf{s}_i$
  - An example: $\mathbf{Q}_1 = \mathbf{Q}_2$ means that $\mathbf{d}_j = \mathbf{s}_i$

T indicates the transposition

Vector Cross Product

- Let $\mathbf{x} = [x_1, x_2, x_3]^\mathsf{T}$ and $\mathbf{y} = [y_1, y_2, y_3]^\mathsf{T}$ denote two vectors
- The vector cross-product is the vector $\mathbf{z} = \mathbf{x} \times \mathbf{y}$ being orthogonal to both $\mathbf{x}$ and $\mathbf{y}$, i.e. $\mathbf{x}^\mathsf{T}\mathbf{z} = \mathbf{y}^\mathsf{T}\mathbf{z} = 0$

$$\mathbf{x} \times \mathbf{y} = \begin{bmatrix} x_2 y_3 - x_3 y_2 & -x_1 y_3 + x_3 y_1 & x_1 y_2 - x_2 y_1 \end{bmatrix}^\mathsf{T}$$

$$= \begin{bmatrix} 0 & -x_3 & x_2 \\ x_3 & 0 & -x_1 \\ -x_2 & x_1 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}^\mathsf{T} \begin{bmatrix} 0 & -y_3 & y_2 \\ y_3 & 0 & -y_1 \\ -y_2 & y_1 & 0 \end{bmatrix}$$

  – An example: if $\mathbf{x}$ and $\mathbf{y}$ are homogeneous coordinates of two 2D points then a 2D line through these points has the parameter vector, $\mathbf{a} = [a_1, a_2, a_3]^\mathsf{T}$, such that $\mathbf{a} = \mathbf{x} \times \mathbf{y}$

T indicates the transposition

Fundamental Matrix

- Epipoles: $\mathbf{e}_1 = \mathbf{P}_1\ [(-\mathbf{Q}_2^{-1}\mathbf{q}_2)^T\ \ 1]^T$ $\quad \mathbf{e}_2 = \mathbf{P}_2\ [(-\mathbf{Q}_1^{-1}\mathbf{q}_1)^T\ \ 1]^T$
- Image points at infinity of the optical ray:

$$\mathbf{d}_1 = \mathbf{Q}_1\mathbf{Q}_2^{-1}\,\mathbf{s}_2 \qquad\qquad \mathbf{d}_2 = \mathbf{Q}_2\mathbf{Q}_1^{-1}\,\mathbf{s}_1$$

- Given the two points $\mathbf{e}_1$ and $\mathbf{d}_1$ ($\mathbf{e}_2$ and $\mathbf{d}_2$), the epipolar line in the image plane 1 (2) in homogeneous coordinates is given by the *vector cross product* : $\mathbf{e}_1$ x $\mathbf{d}_1$ ($\mathbf{e}_2$ x $\mathbf{d}_2$)
- It follows that the cross products $\mathbf{e}_1$ x $\mathbf{d}_1$ and $\mathbf{e}_2$ x $\mathbf{d}_2$ can be written as $\mathbf{s}_2^T\mathbf{F}$ and $\mathbf{F}\mathbf{s}_1$, respectively
- $\mathbf{F}$ is a 3 x 3 **fundamental matrix**
- Any pixel $\mathbf{s}_1$ ($\mathbf{s}_2$) on the epipolar line of $\mathbf{s}_2$ ($\mathbf{s}_1$) satisfies the Longuet-Higgins equation: $\mathbf{s}_2^T\mathbf{F}\ \mathbf{s}_1 = 0$

T indicates the transposition

Fundamental Matrix

- The parameters of the epipolar line of $\mathbf{s}_1$ are given by the vector $\mathbf{s}_2^{\mathsf{T}}\mathbf{F}$ as well as the parameters of the epipolar line of $\mathbf{s}_2$ - by the vector $\mathbf{F}\mathbf{s}_1$
  - Let homogeneous coordinate vectors $\mathbf{s}_{k,j}=[x_{k,j}\ y_{k,j}\ 1]^{\mathsf{T}}$; $j=1,2$ denote the k-th pair of corresponding points in a stereo pair of images
  - The indices $j=1$ and 2 represent the left and right image of the stereo pair, respectively
- Meaning of the fundamental matrix relationship: $\mathbf{s}_{k,2}^{\mathsf{T}}\mathbf{F}\ \mathbf{s}_{k,1} = 0$
  - Any point $\mathbf{s}_{k,2}$ of the right image specifies in the left image an epipolar line which the corresponding point $\mathbf{s}_{k,1}$ lies on; the line parameters are $\mathbf{s}_{k,2}^{\mathsf{T}}\mathbf{F}$
  - Alternatively, a point $\mathbf{s}_{k,1}$ specifies in the right image the parameters $\mathbf{F}\mathbf{s}_{k,1}$ of the corresponding epipolar line which the point $\mathbf{s}_{k,2}$ lies on

Fundamental Matrix

- Parameters of the epipolar lines are represented by the coordinates of the epipoles $\mathbf{e}_1=[x_{e,1}\ y_{e,1}]^T$ and $\mathbf{e}_2=[x_{e,2}\ y_{e,2}]^T$:

$$
\begin{aligned}
& a_1(x_{k,1}-x_{e,1})(x_{k,2}-x_{e,2}) \\
+\, & a_2(y_{k,1}-y_{e,1})(y_{k,2}-y_{e,2}) \\
+\, & a_4(x_{k,1}-x_{e,1})(y_{k,2}-y_{e,2}) \\
+\, & a_5(y_{k,1}-y_{e,1})(x_{k,2}-x_{e,2}) \quad = 0
\end{aligned}
$$

Fundamental Matrix

- The fundamental matrix depends on four parameters

$$\mathbf{a} = [a_1, a_2, a_4, a_5]^\mathrm{T}$$

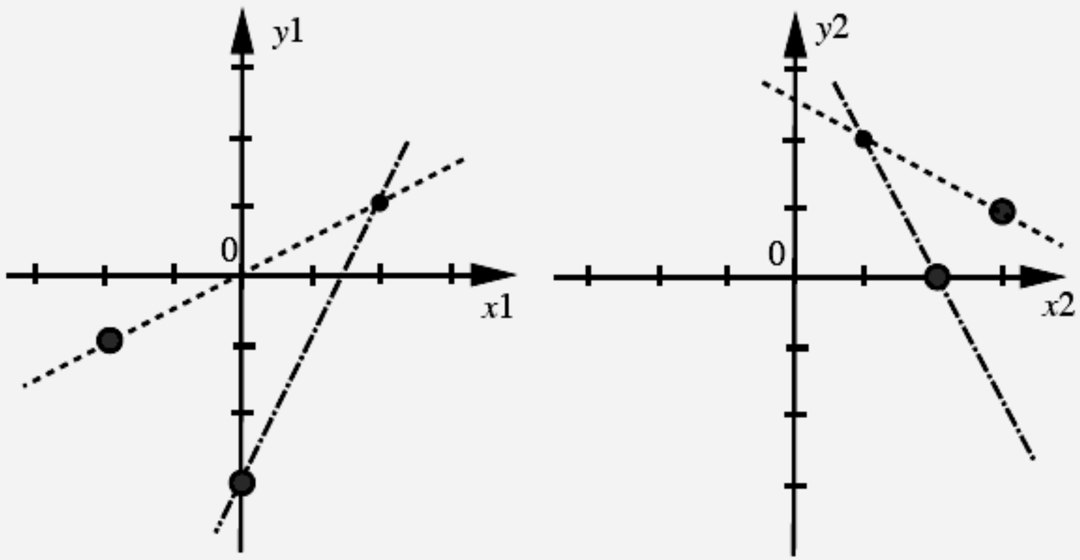
and four coordinates

$$\mathbf{e} = [x_{e,1}, y_{e,1}, x_{e,2}, y_{e,2}]^\mathrm{T}$$

of the epipoles:

$$\mathbf{F} = \begin{pmatrix} a_1 & a_2 & -x_{e,1}a_1 - y_{e,1}a_2 \\ a_4 & a_5 & -x_{e,1}a_4 - y_{e,1}a_5 \\ -x_{e,2}a_1 - y_{e,2}a_4 & -x_{e,2}a_2 - y_{e,2}a_5 & \begin{matrix} x_{e,1}x_{e,2}a_1 + y_{e,1}x_{e,2}a_2 \\ + x_{e,1}y_{e,2}a_4 + y_{e,1}y_{e,2}a_5 \end{matrix} \end{pmatrix}$$

- It is easily seen that the fundamental matrix has the rank 2



$$F = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & -2 \\ -2 & -1 & 5 \end{bmatrix}$$

epipolar line 1: coefficients $y_2 - 2,\ x_2 - 1,\ -x_2 - 2y_2 + 5$

epipolar line 2: coefficients $y_1 - 1,\ x_1 - 2,\ -2x_1 - y_1 + 5$

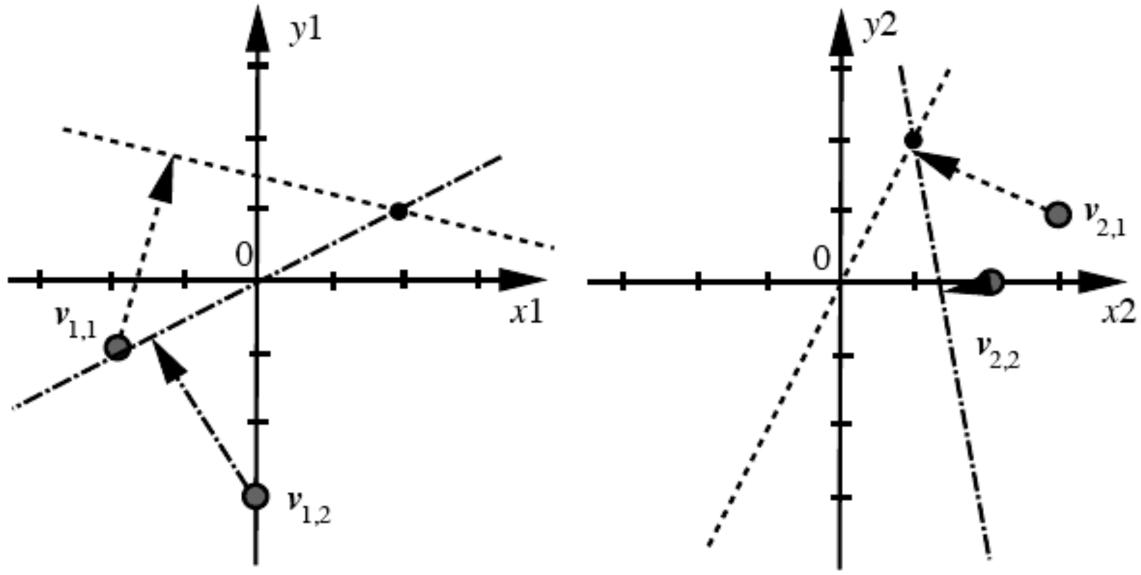
epipole 1: $\mathbf{e}_1 = [\,2, 1, 1\,]^T$

epipole 2: $\mathbf{e}_2 = [\,1, 2, 1\,]^T$

corresponding pixels: $\mathbf{x}_1 = [\,-2, -1, 1\,]$: $\mathbf{coeffs}_1\ (-2, -4, 10)$
$\mathbf{x}_2 = [\,3, 1, 1\,]$: $\mathbf{coeffs}_2\ (-1, 2, 0)$

corresponding pixels: $\mathbf{x}_1 = [\,0, -3, 1\,]$: $\mathbf{coeffs}_1\ (-4, -2, 8)$
$\mathbf{x}_2 = [\,2, 0, 1\,]$: $\mathbf{coeffs}_2\ (-2, 1, 3)$

Corresponding pixels $\mathbf{s}_{j,k}$; $j=1,2$; $k=1,2$, and the epipolar lines for given parameters $\mathbf{a}$, $\mathbf{e}$



$$\mathbf{F} = \begin{bmatrix} 1 & 2 & -4 \\ 1 & 0 & -2 \\ -3 & -2 & 8 \end{bmatrix}$$

ep. line 1: coeffs $x_2 + y_2 - 3,\ 2x_2 - 2,\ -4x_2 - 2y_2 + 8$

ep. line 2: coeffs $x_1 + 2y_1 - 4,\ x_1 - 2,\ -3x_1 - 2y_1 + 8$

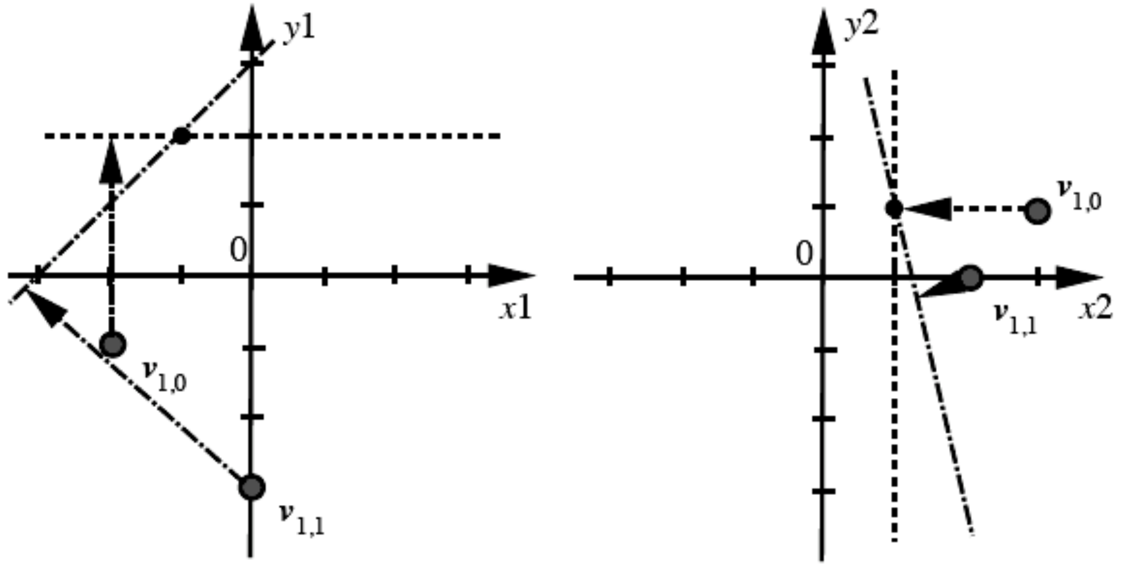
epipole 1: $\mathbf{e}_1 = [\,2, 1, 1\,]^T$

epipole 2: $\mathbf{e}_2 = [\,1, 2, 1\,]^T$

corresponding pixels: $\mathbf{x}_1 = [\,-2, -1, 1\,]$: **coeffs**$_2$ $(-8, -4, 16)$
$\mathbf{x}_2 = [\,3, 1, 1\,]$: **coeffs**$_1$ $(1, 4, -6)$

corresponding pixels: $\mathbf{x}_1 = [\,0, -3, 1\,]$: **coeffs**$_2$ $(-10, -2, 14)$
$\mathbf{x}_2 = [\,2, 0, 1\,]$: **coeffs**$_1$ $(-1, 2, 0)$

Changes of the epipolar lines for new parameters $\mathbf{a}$



$$F = \begin{bmatrix} 0 & 1 & -2 \\ 1 & 0 & 1 \\ -1 & -1 & 1 \end{bmatrix}$$

epipolar line 1: coefficients $y_2 - 1,\ x_2 - 1,\ -2x_2 + y_2 + 1$

epipolar line 2: coefficients $y_1 - 2,\ x_1 + 1,\ -x_1 - y_1 + 1$

epipole 1: $\mathbf{e}_1 = [\,-1, 2, 1\,]^T$

epipole 2: $\mathbf{e}_2 = [\,1, 1, 1\,]^T$

corresponding pixels: $\mathbf{x}_1 = [\,-2, -1,\ 1\,]$: $\mathbf{coeffs}_1\ (\ 0,\ \ 2,\ -4\ )$
$\mathbf{x}_2 = [\ \ 3,\ \ 1,\ 1\,]$: $\mathbf{coeffs}_2\ (\,-3,\ \ 0,\ \ 3\ )$

corresponding pixels: $\mathbf{x}_1 = [\ 0,\ -3,\ \ 1\,]$: $\mathbf{coeffs}_1\ (\,-1,\ \ 1, -3\ )$
$\mathbf{x}_2 = [\ 2,\ \ 0,\ \ 1\,]$: $\mathbf{coeffs}_2\ (\,-5,\ \ 1,\ \ 4\ )$

Changes of the epipolar lines for new positions of the epipoles $\mathbf{e}$

Distance to an Epipolar Line

- The unnormalised (or scaled) squared distance between a pixel an an epipolar line generated by the corresponding pixel can be represented as follows:

$$(\mathbf{s}_{k,2}^{\mathsf{T}}\mathbf{F}\,\mathbf{s}_{k,1})^2 \equiv \mathbf{a}^{\mathsf{T}}\boldsymbol{\Phi}_k(\mathbf{e})\mathbf{a}$$

with the following 4 x 4 matrix $\boldsymbol{\Phi}_k(\mathbf{e}) = \mathbf{f}_k(\mathbf{e})\,\mathbf{f}_k^{\mathsf{T}}(\mathbf{e})$ where the vector $\mathbf{f}_k(\mathbf{e})$ combines the coordinates of the corresponding pixels and the epipoles:

$$\mathbf{f}_k(\mathbf{e}) = \begin{bmatrix} (x_{k,1} - x_{\mathrm{e},1})(x_{k,2} - x_{\mathrm{e},2}) \\ (y_{k,1} - y_{\mathrm{e},1})(x_{k,2} - x_{\mathrm{e},2}) \\ (x_{k,1} - x_{\mathrm{e},1})(y_{k,2} - y_{\mathrm{e},2}) \\ (y_{k,1} - y_{\mathrm{e},1})(y_{k,2} - y_{\mathrm{e},2}) \end{bmatrix}$$

Distance to an Epipolar Line

- Therefore the squared distance $d_{k,1}(\mathbf{a},\mathbf{e})$ of a pixel $\mathbf{s}_{k,1}$ from the epipolar line that corresponds to the pixel $\mathbf{s}_{k,2}$, and the like distance $d_{k,2}(\mathbf{a},\mathbf{e})$ of the pixel $\mathbf{s}_{k,2}$ from the epipolar line that corresponds to the pixel $\mathbf{s}_{k,1}$ are as follows:

$$d_{k,1}(\mathbf{a},\mathbf{e}) = \frac{\mathbf{a}^{\mathsf{T}}\Phi_k(\mathbf{e})\mathbf{a}}{\mathbf{a}^{\mathsf{T}}\mathbf{C}_{k,1}(\mathbf{e}_2)\mathbf{a}}; \quad d_{k,2}(\mathbf{a},\mathbf{e}) = \frac{\mathbf{a}^{\mathsf{T}}\Phi_k(\mathbf{e})\mathbf{a}}{\mathbf{a}^{\mathsf{T}}\mathbf{C}_{k,2}(\mathbf{e}_1)\mathbf{a}}$$

  - The denominators are the normalising factors:

$$\mathbf{a}^{\mathsf{T}}\mathbf{C}_{k,1}(\mathbf{e}_2)\mathbf{a} \equiv \left(a_1 \cdot \left(x_{k,2} - x_{\mathrm{e},2}\right) + a_2 \cdot \left(y_{k,2} - y_{\mathrm{e},2}\right)\right)^2 + \left(a_4 \cdot \left(x_{k,2} - x_{\mathrm{e},2}\right) + a_5 \cdot \left(y_{k,2} - y_{\mathrm{e},2}\right)\right)^2$$

$$\mathbf{a}^{\mathsf{T}}\mathbf{C}_{k,2}(\mathbf{e}_1)\mathbf{a} \equiv \left(a_1 \cdot \left(x_{k,1} - x_{\mathrm{e},1}\right) + a_4 \cdot \left(y_{k,1} - y_{\mathrm{e},1}\right)\right)^2 + \left(a_2 \cdot \left(x_{k,1} - x_{\mathrm{e},1}\right) + a_5 \cdot \left(y_{k,1} - y_{\mathrm{e},1}\right)\right)^2$$

Normalising factors

$$\mathbf{C}_{k,1}(\mathbf{e}_2) = \begin{pmatrix} (x_{k,2}-x_{e,2})^2 & (x_{k,2}-x_{e,2})\cdot(y_{k,2}-y_{e,2}) & 0 & 0 \\ (x_{k,2}-x_{e,2})\cdot(y_{k,2}-y_{e,2}) & (y_{k,2}-y_{e,2})^2 & 0 & 0 \\ 0 & 0 & (x_{k,2}-x_{e,2})^2 & (x_{k,2}-x_{e,2})\cdot(y_{k,2}-y_{e,2}) \\ 0 & 0 & (x_{k,2}-x_{e,2})\cdot(y_{k,2}-y_{e,2}) & (y_{k,2}-y_{e,2})^2 \end{pmatrix}$$

$$\mathbf{C}_{k,2}(\mathbf{e}_1) = \begin{pmatrix} (x_{k,1}-x_{e,1})^2 & 0 & (x_{k,1}-x_{e,1})\cdot(y_{k,1}-y_{e,1}) & 0 \\ 0 & (x_{k,1}-x_{e,1})^2 & 0 & (x_{k,1}-x_{e,1})(y_{k,1}-y_{e,1}) \\ (x_{k,1}-x_{e,1})\cdot(y_{k,1}-y_{e,1}) & 0 & (y_{k,1}-y_{e,1})^2 & 0 \\ 0 & (x_{k,1}-x_{e,1})\cdot(y_{k,1}-y_{e,1}) & 0 & (y_{k,1}-y_{e,1})^2 \end{pmatrix}$$

Normalised Fundamental Matrix

- Components of the fundamental matrix have to be normalised to exclude the singular case of $\mathbf{F} = \mathbf{0}$
- An ideal horizontal stereo pair with the epipolar lines $y_1 = y_2 = y$ that are parallel to the x-axis of the images has the following fundamental matrix:

$$\mathbf{F} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 0 \end{bmatrix}$$

- Here, the parameters $\mathbf{a} = \mathbf{0}$ and the parameters $\mathbf{e} = [-\infty, c_1, \infty, c_2]^{\mathsf{T}}$ where the constants c_j may have arbitrary values
  - It is impossible to normalise only the parameters $\mathbf{a}$: all the components which are present in the normalising factors for the distance should be taken into account, i.e. all the components of $\mathbf{F} = [f_{ij}]_{ij=1,2,3}$ excepting the component $f_{3,3}$

Fundamental Matrix: Computation

- Given a large set of corresponding points $\{(\mathbf{s}_{1,k}, \mathbf{s}_{2,k}): i = 1,\ldots,n\}$, the equation $\mathbf{s}_1^{\mathrm{T}}\mathbf{F}\mathbf{s}_2=0$ can be used to estimate $\mathbf{F}$
- Each point match $\mathbf{s}_{1,k}=[x_{1,k},y_{1,k},1]^{\mathrm{T}}$ and $\mathbf{s}_{2,k}=[x_{2,k},y_{2,k},1]^{\mathrm{T}}$ results in one linear equation for the unknown entries of $\mathbf{F}$:

$$\begin{aligned} & x_{1,k}x_{2,k}f_{11} + x_{1,k}y_{2,k}f_{12} + x_{1,k}f_{13} \\ & + y_{1,k}x_{2,k}f_{21} + y_{1,k}y_{2,k}f_{22} + y_{1,k}f_{23} \\ & + x_{2,k}f_{31} + y_{2,k}f_{32} + f_{33} \qquad = 0, \text{ or} \end{aligned}$$

$$[x_{1,k}x_{2,k}, x_{1,k}y_{2,k}, x_{1,k}, y_{1,k}x_{2,k}, y_{1,k}x_{2,k}, y_{1,k}, x_{2,k}, y_{2,k}, 1]\mathbf{f} = 0$$

- For a set of n point matches: a set of linear equations:

$$\mathbf{A}\mathbf{f} = \begin{bmatrix} x_{1,1}x_{2,1}, x_{1,1}y_{2,1}, x_{1,1}, y_{1,1}x_{2,1}, y_{1,1}y_{2,1}, y_{1,1}, x_{2,1}, y_{2,1}, 1 \\ \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\ x_{1,n}x_{2,n}, x_{1,n}y_{2,n}, x_{1,n}, y_{1,n}x_{2,n}, y_{1,n}y_{2,n}, y_{1,n}, x_{2,n}, y_{2,n}, 1 \end{bmatrix}\mathbf{f} = \mathbf{0}$$

Fundamental Matrix: Computation

- The set of equations $\mathbf{Af=0}$ is homogeneous: so $\mathbf{f}$ can be determined up to scale
  - For a solution to exist, $\mathbf{A}$ should have rank at most 8
  - If the rank is exactly 8, then the solution is unique (up to scale), and can be found by linear methods
- A least-squares solution for noisy data: $\min_{\mathbf{f}} \|\mathbf{Af}\|$
  - The data are not exact (noisy) and the rank of $\mathbf{A}$ is greater than 8 (i.e. equal to 9 because $\mathbf{A}$ has 9 columns)
  - The least-squares solution for $\mathbf{f}$ is the singular vector corresponding to the smallest singular value of $\mathbf{A}$, i.e. the last column of the matrix $\mathbf{V}$ in the singular value decomposition (SVD) $\mathbf{A} = \mathbf{UDV}^T$
    - The solution vector $\mathbf{f}$ found in this way minimises the vector norm $\|\mathbf{Af}\|$ subject to the condition $\|\mathbf{f}\|=1$
    - The singularity constraint: the fundamental matrix $\mathbf{F}$ has rank 2

The 8-Point Algorithm

- Enforcing the singularity constraint by correcting the matrix $\mathbf{F}$ found by the SVD solution from $\mathbf{A}$
  - Close approximation of $\mathbf{F}$ with the matrix $\mathbf{F'}$ with zero determinant $|\mathbf{F'}| = 0$
  - Can be done by the SVD: if $\mathbf{F}=\mathbf{UDV}^T$ is the SVD of $\mathbf{F}$ where $\mathbf{D}$ is the diagonal matrix $\mathbf{D}=\mathrm{diag}\{\alpha,\beta,\gamma\}$ such that $\alpha\geq\beta\geq\gamma$, then $\mathbf{F'}=\mathbf{U}\mathrm{diag}\{\alpha,\beta,0\}\mathbf{V}^T$
- **The normalised 8-point algorithm**
  - ***Initial normalisation*** of input data: translation and scaling of each image so that the centroid of reference points is at the origin of the coordinates and the root mean square (RMS) distance of the points from the origin is equal to $\sqrt{2}$
  - (*i*) ***Linear solution*** $\mathbf{F}$ is obtained from the vector $\mathbf{f}$ corresponding to the minimal singular value of $\mathbf{A}$ specifying the system of equations $\mathbf{Af} = \mathbf{0}$
  - (*ii*) **Singularity constraint** is enforced by replacing $\mathbf{F}$ by $\mathbf{F'}$, the closest singular matrix to $\mathbf{F}$, using the SVD
  - ***Denornalisation***: the linear transformation of $\mathbf{F'}$ to fit the non-normalised data

Singular Value Decomposition

- Any generic m x n rectangular matrix A can be written as the product of three matrices: $\mathbf{A} = \mathbf{UDV}^T$
  - The columns of the m x m matrix $\mathbf{U}$ are mutually orthogonal unit vectors
  - The columns of the n x n matrix $\mathbf{V}$ are mutually orthogonal unit vectors
  - The m x n diagonal matrix $\mathbf{D}$ has diagonal elements σ_i called singular values such that $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_N \geq 0$ $(N = \min\{m,n)\})$
  - The matrices $\mathbf{U}$ and $\mathbf{V}$ are not unique, but the singular values are fully determined by the matrix $\mathbf{A}$
- A square matrix $\mathbf{A}$ is non-singular if and only if all its singular values are different from zero
  - Ratio $C = \sigma_1 / \sigma_n$ (condition number) - the degree of singularity of $\mathbf{A}$
    - If $1/C$ is comparable with the arithmetic precision of a computer, the matrix $\mathbf{A}$ is ill-conditioned and for all practical purposes should be considered singular

Singular Value Decomposition

- If $\mathbf{A}$ is a rectangular matrix, the number of non-zero singular values σ_i equals the rank of A
  - Given a fixed tolerance, ε, being typically of order 10^{-6}, the number of singular values greater than ε equals the effective rank of $\mathbf{A}$
- If $\mathbf{A}$ is a square, non-singular matrix, its inverse $\mathbf{A}^{-1} = \mathbf{V}\mathbf{D}^{-1}\mathbf{U}^{\mathsf{T}}$
  - Be $\mathbf{A}$ singular or not, the **pseudoinverse** of $\mathbf{A}$, $\mathbf{A}^{+}$, is $\mathbf{A}^{+} = \mathbf{V}\mathbf{D}_0^{-1}\mathbf{U}^{\mathsf{T}}$
  - $\mathbf{D}_0^{-1}$ is equal to $\mathbf{D}^{-1}$ for all nonzero singular values and zero otherwise
  - If $\mathbf{A}$ is nonsingular, then $\mathbf{D}_0^{-1} = \mathbf{D}^{-1}$ and $\mathbf{A}^{+} = \mathbf{A}^{-1}$
- The columns of $\mathbf{U}$ are eigenvectors of $\mathbf{A}\mathbf{A}^{\mathsf{T}}$
- The columns of $\mathbf{V}$ are eigenvectors of $\mathbf{A}^{\mathsf{T}}\mathbf{A}$

Singular Value Decomposition

- Property of the SVD: $\mathbf{A}v_i = \sigma_i u_i$ and $\mathbf{A}^\mathsf{T} u_i = \sigma_i v_i$
  - Here, u_i and v_i are the columns of $\mathbf{U}$ and $\mathbf{V}$ corresponding to σ_i
- The squares of the nonzero singular values are the nonzero eigen-values of both the n x n matrix $\mathbf{A}^\mathsf{T}\mathbf{A}$ and m x m matrix $\mathbf{A}\mathbf{A}^\mathsf{T}$
- There is another definition of SVD:

  with the m x n matrix $\mathbf{U}$ and n x n matrices $\mathbf{D}$ and $\mathbf{V}$
- The latter definition is typically used in computations because of a smaller memory space for the matrices: $mn + 2N^2$ rather that $m^2 + mn + N^2$ for the initial definition as typically $N << m$

SVD: An Example

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow AA^{\mathsf{T}} = A^{\mathsf{T}}A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \Rightarrow \text{ Eigenvalues and eigenvectors for } AA^{\mathsf{T}} = A^{\mathsf{T}}A:$$

$$\lambda_{1,2} = 1;\ e_1 = u_1 = v_1 = \begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix};\ e_2 = u_2 = v_2 = \begin{bmatrix} 0 \\ -\sin\theta \\ \cos\theta \end{bmatrix};;\ \lambda_3 = 0;\ e_3 = u_3 = v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$$

$$Av_i = \sigma_i u_i;\quad i = 1,2,3 \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix} = \sigma_1 \begin{bmatrix} 0 \\ \cos\theta \\ \sin\theta \end{bmatrix} \Rightarrow \sigma_1 = 1;\ \cos\theta = \sin\theta = \frac{1}{\sqrt{2}};$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \sigma_2 \begin{bmatrix} 0 \\ -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \Rightarrow \sigma_2 = -1;\qquad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \sigma_3 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \sigma_3 = 0 \Rightarrow$$

$$A = \underbrace{\begin{bmatrix} e_1 & e_2 & e_3 \end{bmatrix}}_{U} \underbrace{\text{diag}\{\sigma_1,\sigma_2,\sigma_3\}}_{D} \underbrace{\begin{bmatrix} e_1^{\mathsf{T}} \\ e_2^{\mathsf{T}} \\ e_3^{\mathsf{T}} \end{bmatrix}}_{V^{\mathsf{T}}} = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \end{bmatrix}}_{V^{\mathsf{T}}}$$

SVD: An Example

$$A=\begin{bmatrix}0&1\\1&1\\1&0\end{bmatrix}\Rightarrow AA^{\mathsf T}=\begin{bmatrix}0&1\\1&1\\1&0\end{bmatrix}\begin{bmatrix}0&1&1\\1&1&0\end{bmatrix}=\begin{bmatrix}1&1&0\\1&2&1\\0&1&1\end{bmatrix};\quad A^{\mathsf T}A=\begin{bmatrix}0&1&1\\1&1&0\end{bmatrix}\begin{bmatrix}0&1\\1&1\\1&0\end{bmatrix}=\begin{bmatrix}2&1\\1&2\end{bmatrix}\Rightarrow$$

Eigenvalues and eigenvectors for $AA^{\mathsf T}$: $(1-\lambda)[(2-\lambda)(1-\lambda)-1]-(1-\lambda)=(3-\lambda)(1-\lambda)\lambda=0\Rightarrow$

$$\lambda_1=3;\ u_1=\begin{bmatrix}1/\sqrt6\\2/\sqrt6\\1/\sqrt6\end{bmatrix};\ \lambda_2=1;\ u_2=\begin{bmatrix}1/\sqrt2\\0\\-1/\sqrt2\end{bmatrix};\ \lambda_3=0;\ u_3=\begin{bmatrix}1/\sqrt3\\-1/\sqrt3\\1/\sqrt3\end{bmatrix};$$

Eigenvalues and eigenvectors for $A^{\mathsf T}A$: $(2-\lambda)^2-1=(3-\lambda)(1-\lambda)=0\Rightarrow$

$$\lambda_1=3;\ v_1=\begin{bmatrix}1/\sqrt2\\1/\sqrt2\end{bmatrix};\ \lambda_2=1;\ v_2=\begin{bmatrix}1/\sqrt2\\-1/\sqrt2\end{bmatrix};\ \Rightarrow$$

$$Av_i=\sigma_iu_i;\ i=1,2\ \Rightarrow\begin{bmatrix}0&1\\1&1\\1&0\end{bmatrix}\begin{bmatrix}1/\sqrt2\\1/\sqrt2\end{bmatrix}=\sigma_1\begin{bmatrix}1/\sqrt6\\2/\sqrt6\\1/\sqrt6\end{bmatrix}\Rightarrow\sigma_1=\sqrt3;\quad\begin{bmatrix}0&1\\1&1\\1&0\end{bmatrix}\begin{bmatrix}1/\sqrt2\\-1/\sqrt2\end{bmatrix}=\sigma_2\begin{bmatrix}1/\sqrt2\\0\\-1/\sqrt2\end{bmatrix}\Rightarrow\sigma_2=-1$$

$$\Rightarrow\quad A=\underbrace{[u_1\quad u_2\quad u_3]}_{U}\underbrace{\mathrm{diag}\{\sigma_1,\sigma_2\}}_{D}\underbrace{\begin{bmatrix}e_1^{\mathsf T}\\e_2^{\mathsf T}\end{bmatrix}}_{V^{\mathsf T}}=\underbrace{\begin{bmatrix}1/\sqrt6&1/\sqrt2&1/\sqrt3\\2/\sqrt6&0&-1/\sqrt3\\1/\sqrt6&-1/\sqrt2&1/\sqrt3\end{bmatrix}}_{U}\underbrace{\begin{bmatrix}\sqrt3&0\\0&-1\\0&0\end{bmatrix}}_{D}\underbrace{\begin{bmatrix}1/\sqrt2&1/\sqrt2\\1/\sqrt2&-1/\sqrt2\end{bmatrix}}_{V^{\mathsf T}}$$

Rectification of Stereo Images

- **Rectification** of a stereo pair is a transformation (**warping**) of each image such that pairs of conjugate epipolar lines become collinear and parallel to one of the axes, usually the horizontal one
  - Rectification reduces generally 2D search for correspondence to a 1D search on scan-lines having the same y-coordinate in both the images
- This transformation can be computed using the known intrinsic parameters of each camera and the extrinsic parameters of the stereo system
  - The rectified images can be thought of as acquired by a new stereo rig obtained by rotating the original cameras around their optical centres

Rectification of Stereo Images

- Without losing generality, let us assume that in both cameras:
  - (*i*) the origin of the image reference frame is the principal point (i.e. the trace of the optical axis), and
  - (*ii*) the focal length is equal to f
  - (*iii*) $\mathbf{T}$ and $\mathbf{R}$ are the translation vector ($\mathbf{O}_1\mathbf{O}_2$) and the rotation matrix, respectively, relating the coordinate frames of the left and right cameras
- The rectification algorithm consists in four steps:

1. Rotate the left camera by the rotation matrix $\mathbf{R}_{\mathrm{rect}}$ so that the epipole goes to infinity along the horizontal axis (i.e. the left image plane becomes parallel to the baseline of the system)
2. Apply the same rotation to the right camera to recover the original geometry
3. Rotate the right camera by the rotation matrix $\mathbf{R}$
4. Adjust the scale in both camera reference frames

Rotation Matrix $\mathbf{R}_{\mathbf{rect}}$

$$R_{\text{rect}} = \begin{bmatrix} \mathbf{e}_1^{\mathrm{T}} \\ \mathbf{e}_2^{\mathrm{T}} \\ \mathbf{e}_3^{\mathrm{T}} \end{bmatrix} \text{ where } \mathbf{e}_1 = \frac{\mathbf{T}}{\|\mathbf{T}\|} = \frac{1}{\sqrt{T_x^2 + T_y^2 + T_z^2}} \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix}; \ \mathbf{e}_2 = \frac{\mathbf{e}_1 \times [0,0,1]^{\mathrm{T}}}{\left\|\mathbf{e}_1 \times [0,0,1]^{\mathrm{T}}\right\|} = \frac{1}{\sqrt{T_x^2 + T_y^2}} \begin{bmatrix} -T_y \\ T_x \\ 0 \end{bmatrix};$$

$$\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2 = \frac{1}{\sqrt{\left(T_x^2 + T_y^2\right)\left(T_x^2 + T_y^2 + T_z^2\right)}} \begin{bmatrix} -T_x T_z \\ -T_y T_z \\ T_x^2 + T_y^2 \end{bmatrix}$$

- Partially arbitrary choice of a triple of mutually orthogonal unit vectors $\mathbf{e}$:
- $\mathbf{e}_1$ is given by the epipole (since the image centre is in the origin, the vector $\mathbf{e}_1$ coincides with the direction of translation $\mathbf{T}$)
- $\mathbf{e}_2$ – a vector orthogonal to $\mathbf{e}_1$ (an arbitrary choice: $\mathbf{e}_2 = \mathbf{e}_1$ x **OZ** (the optical axis) before normalisation
- $\mathbf{e}_3 = \mathbf{e}_1$ x $\mathbf{e}_2$
- The remaining steps are straightforward

Rectification Algorithm

Input: the intrinsic and extrinsic parameters of a stereo system;
a set of points in each camera to be rectified (could be the whole images)

Build the rotation matrix $\mathbf{R}_{\text{rect}}$

Set $\mathbf{R}_l = \mathbf{R}_{\text{rect}}$ and $\mathbf{R}_r = \mathbf{R}\mathbf{R}_{\text{rect}}$

for each left-camera point, $\mathbf{p}_l = [x, y, f]^T$,

compute the coordinates of the corresponding rectified point, $\mathbf{p}'_l$, as

$\mathbf{p}'_l = [fx'/z', fy'/z', f]$ where $[x', y', z'] = \mathbf{R}_l \mathbf{p}_l$

Repeat the previous step for the right camera using $\mathbf{R}_r$ and $\mathbf{p}_r$

Output: the pair of transformations to be applied to the two cameras in order to rectify the two input point sets; the rectified sets of points

Rectification of a Stereo Pair