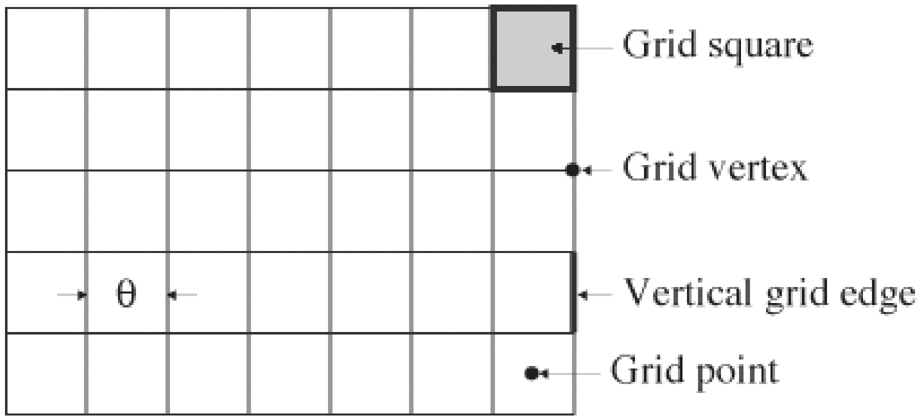
Grid Cells in 2D or 3D

(J. Sklansky 1972, G.T. Herman and D. Webster, 1980)

Grid cubes are 3-cells, grid squares are 2-cells, grid edges are 1-cells, grid vertices are 0-cells (see Lecture 01).



Incidence Relation between Cells

Sets A and B are *incident* iff $A \subseteq B$ or $B \subseteq A$. This relation is reflexive, symmetric, and not transitive.

2D: any m-cell ($m = 0, 1, 2$) is incident with itself, a 2-cell is incident with four 1-cells, a 0-cell is incident with four 1-cells, and so forth

3D: any m-cell ($m = 0, 1, 2, 3$) is incident with itself, a 3-cell is incident with six 2-cells, a 0-cell is incident with six 1-cells, and so forth

Incidence Counts in 2D and 3D

Let a_{ik} be the number of k-cells which are incident with an i-cell.

For $n = 2$, they are as follows:

$$\begin{array}{lll} a_{00} = 1, & a_{01} = 4, & a_{02} = 4, \\ a_{10} = 2, & a_{11} = 1, & a_{12} = 2, \\ a_{20} = 4, & a_{21} = 4, & a_{22} = 1 \end{array}$$

For $n = 3$, they are as follows:

$$\begin{array}{llll} a_{00} = 1, & a_{01} = 6, & a_{02} = 12, & a_{03} = 8, \\ a_{10} = 2, & a_{11} = 1, & a_{12} = 4, & a_{13} = 4, \\ a_{20} = 4, & a_{21} = 4, & a_{22} = 1, & a_{23} = 2, \\ a_{30} = 8, & a_{31} = 12, & a_{32} = 6, & a_{33} = 1 \end{array}$$

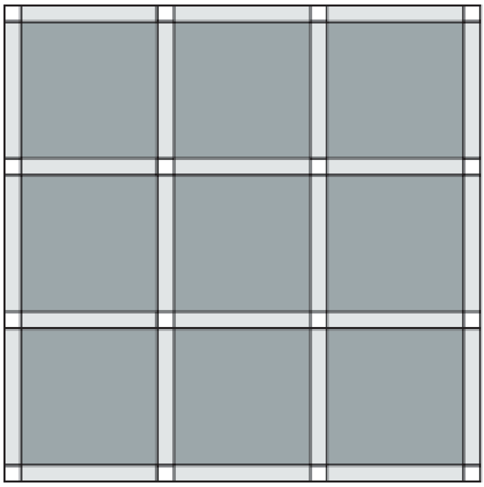
Theorem 1 *In general ($n \geq 1$ and $0 \leq i, j \leq n$):*

$$a_{ij} = \begin{cases} 2^{j-i} \binom{n-i}{n-j} & \text{if } i < j \\ 1 & \text{if } i = j \\ 2^{i-j} \binom{i}{j} & \text{if } i > j \ . \end{cases}$$

(independently published by B.A. Rosenfeld and I.M. Jaglom in 1971, R. Klette in 1972, and H.S.M. Coxeter in 1973)

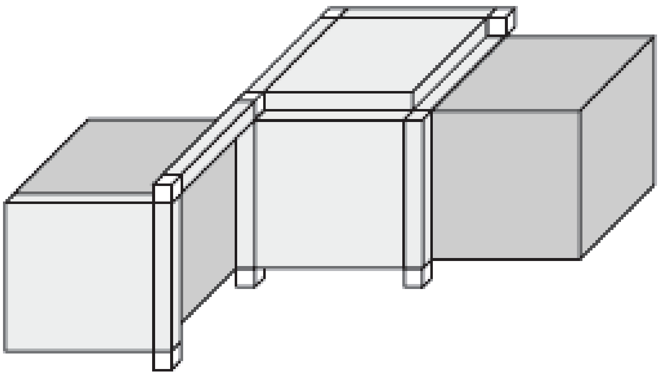
Geometric Representation of Cells

2D: We represent a 2-cell by a large square, a 1-cell by a rectangle, and a 0-cell by a small square.



These squares and rectangles tessellate the infinite plane.

3D: A tessellation of the infinite 3D space (small cubes as 0-cells, elongated cuboids as 1-cells, flat cuboids as 2-cells, and large cubes as 3-cells):



Note: 0- and 1-cells in 2D, or 0-, 1-,and 2-cells in 3D are just *virtual elements*, which can prove to be useful in algorithm design. Still, a picture is given just by values at pixels or voxels.

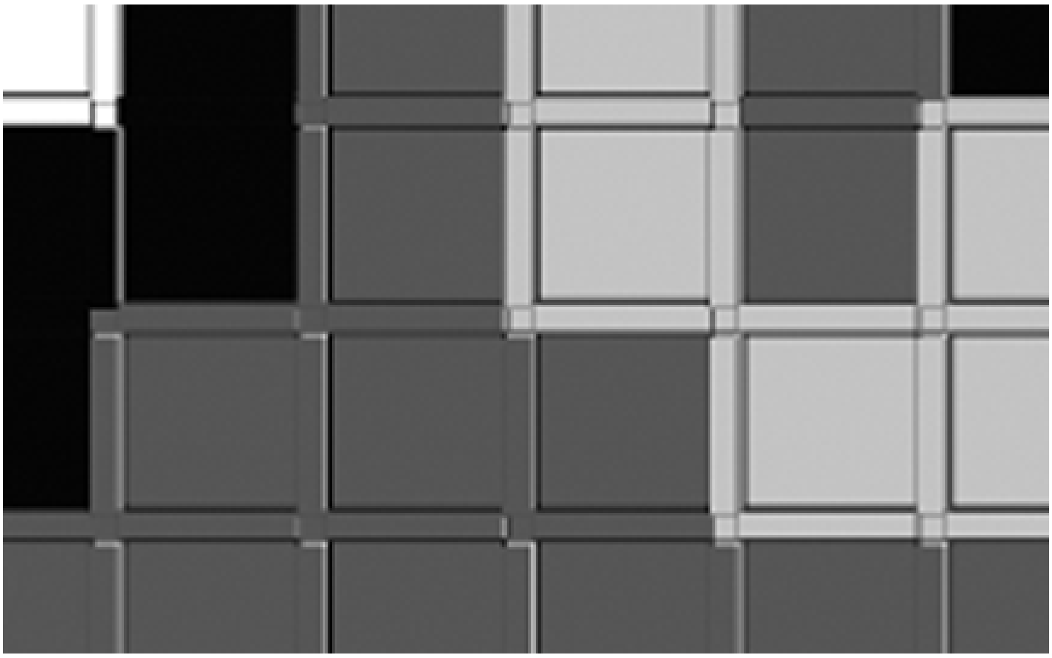
Maximum-Value Rule

(V. Kovalevsky, 1989)

Assume a total order of all possible pixel or voxel values. The generalized rule for 2D ($n = 2$) or 3D ($n = 3$); let $0 \leq m < n$:

> An m-cell (which is incident with a_{mn} n-cells) is assumed to be labeled by the maximum value (in the given total order) at all a_{mn} incident pixels or voxels.

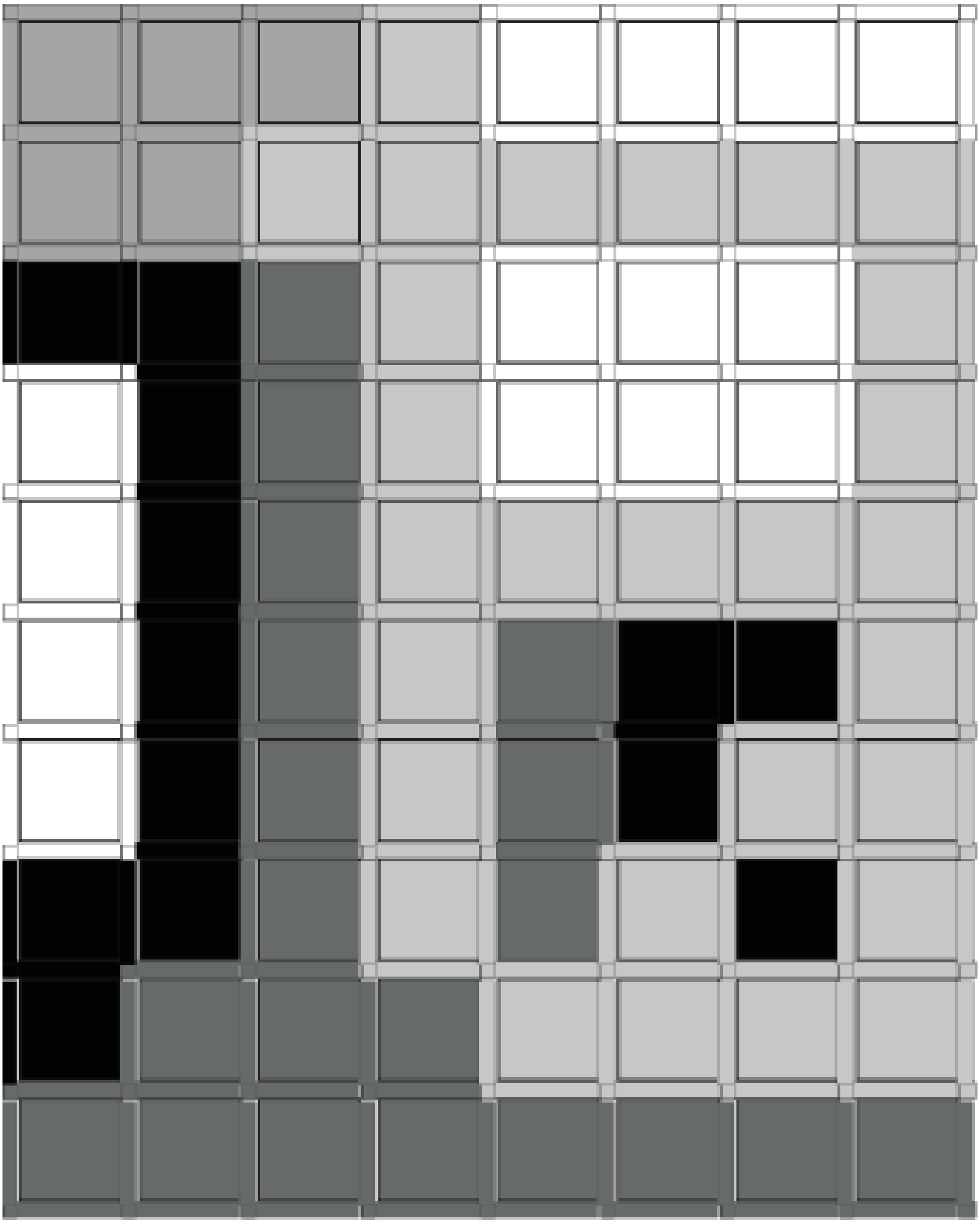
Assume a total order "black < dark gray < light gray < white" .



Pixel values have been assigned to virtual 0- and 1-cells according to the maximum-value rule.

Virtual Values at 0- and 1-Cells

Again: total order “black < dark gray < light gray < white” .

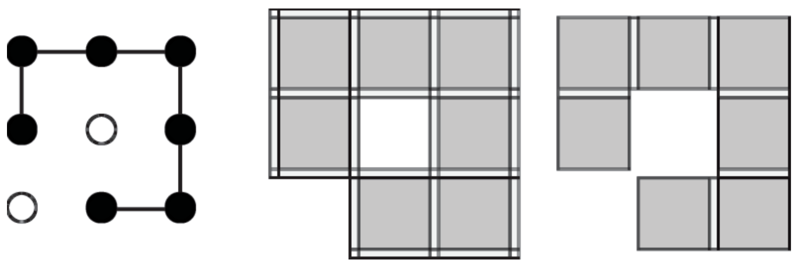


Open and Closed Sets of Pixels

Definition 1 *Let M be a set of n-cells ($n = 2$ in 2D, and $n = 3$ in 3D), which also contains all m-cells ($0 \leq m < n$) which are incident with any of the n-cells in M, and which contains no further cells. Such a set M is called* closed.

The set of all m-cells ($0 \leq m < n$) in a closed set M, which are also incident with at least one n-cell in $\overline{M}$, is called the frontier *of M.*

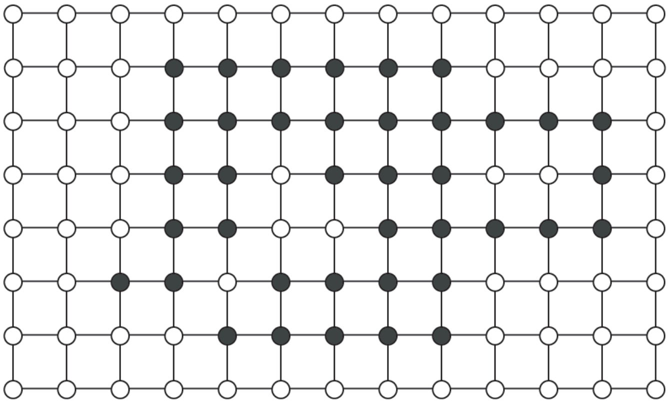
If we remove all frontier cells from a closed set M, then we obtain an open *set.*

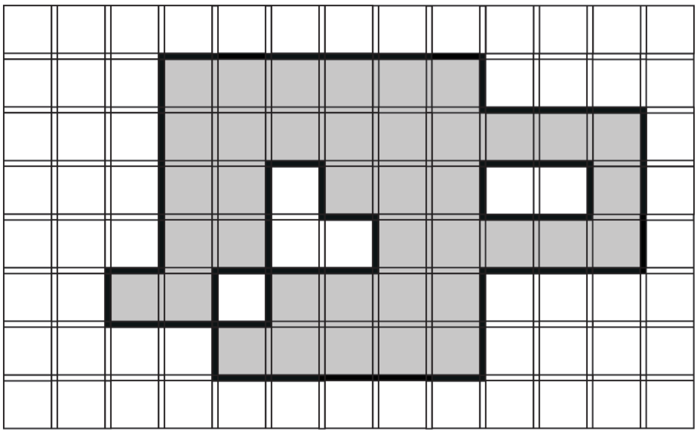


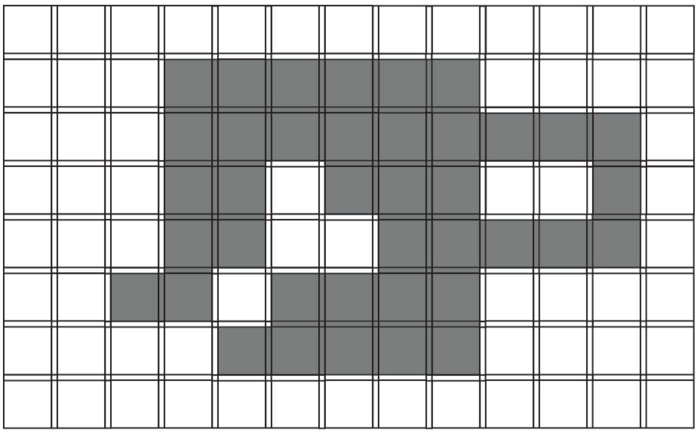
If we add to an open set a few, but not all cells of its frontier, then the resulting set is neither open nor closed.

In the figure on page 5, the white regions are closed (note: "white" is the maximum in the assumed total order), and the black regions are open (note: "black" is the minimum). The light or dark gray regions are neither closed nor open.

Cell Union Represents Connectedness



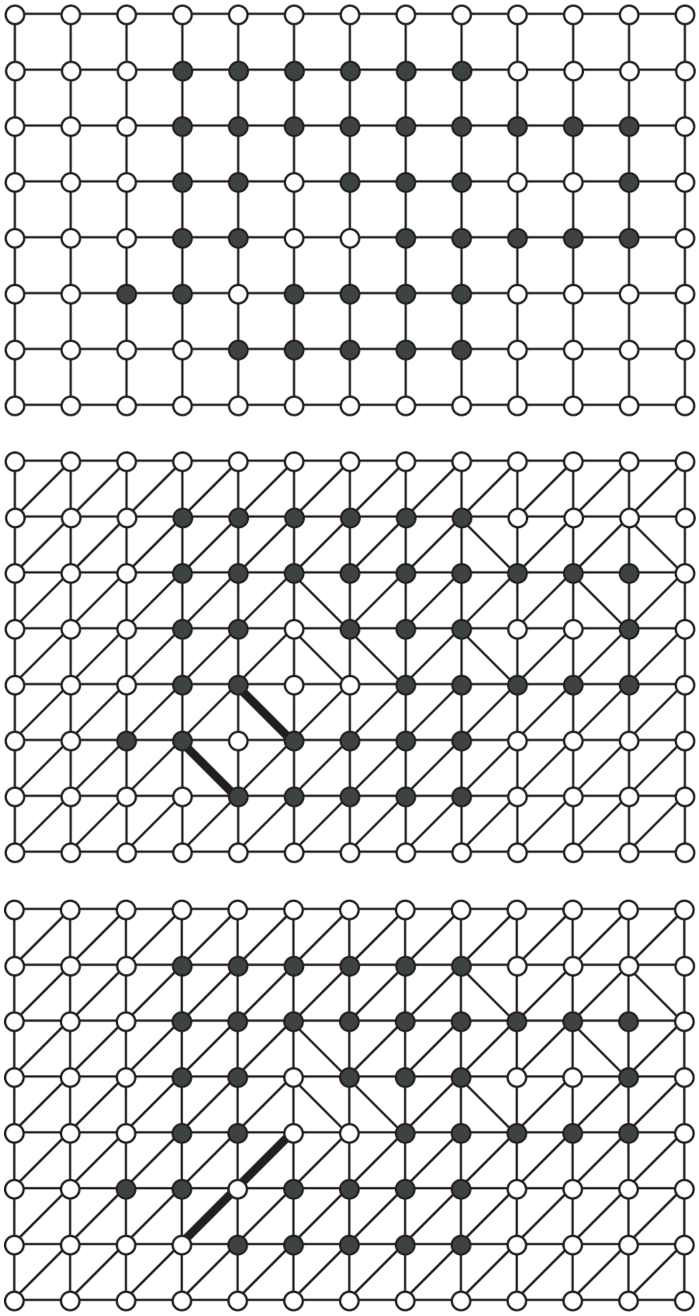




Middle: “white < black”; the black region is closed (frontier: dark 0- and 1-cells); it has three holes

Bottom: “black < white”; the black region is open; has one hole

s-Adjacency



Middle: “white < black”; there are two flip-flop cases, here decided in favor of black

Bottom: “black < white” (inverted flip-flop cases)

Grid Cell Topology

The defined open sets (in 2D or 3D) define a *grid cell topology*. Connectedness in this topology can be represented by unions of cells (and applying standard Euclidean topology to resulting sets).

For a given binary or multilevel (2D or 3D) picture, its topology is decided by an assumed total order of all of its values.

Similarly, s-adjacency was also defined by a total order of all picture values, defining preferences for flip-flop cases.

Second Equivalence Theorem in 2D

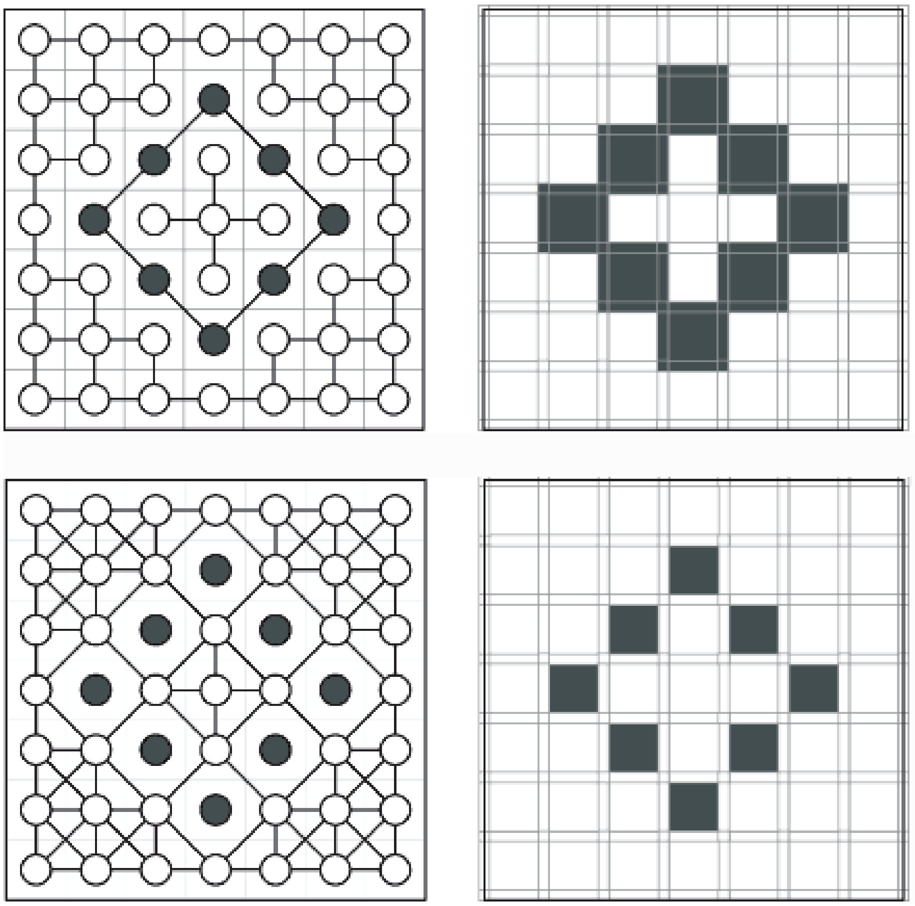
Theorem 2 *For any 2D multilevel picture, the use of grid cell topology can be replaced by s-adjacency (using identical total orders of pixel values) such that the resulting components of pixels are identical for both cases.*

Together with the First Equivalence Theorem, it follows:

For any 2D binary picture, connectedness in the grid cell topology (assuming, e.g., "white $<$ black") is identical to the assumption of 4-adjacency for white pixels, and 8-adjacency for black pixels.

In other words, open sets are realized by 4-adjacency, and closed sets by 8-adjacency.

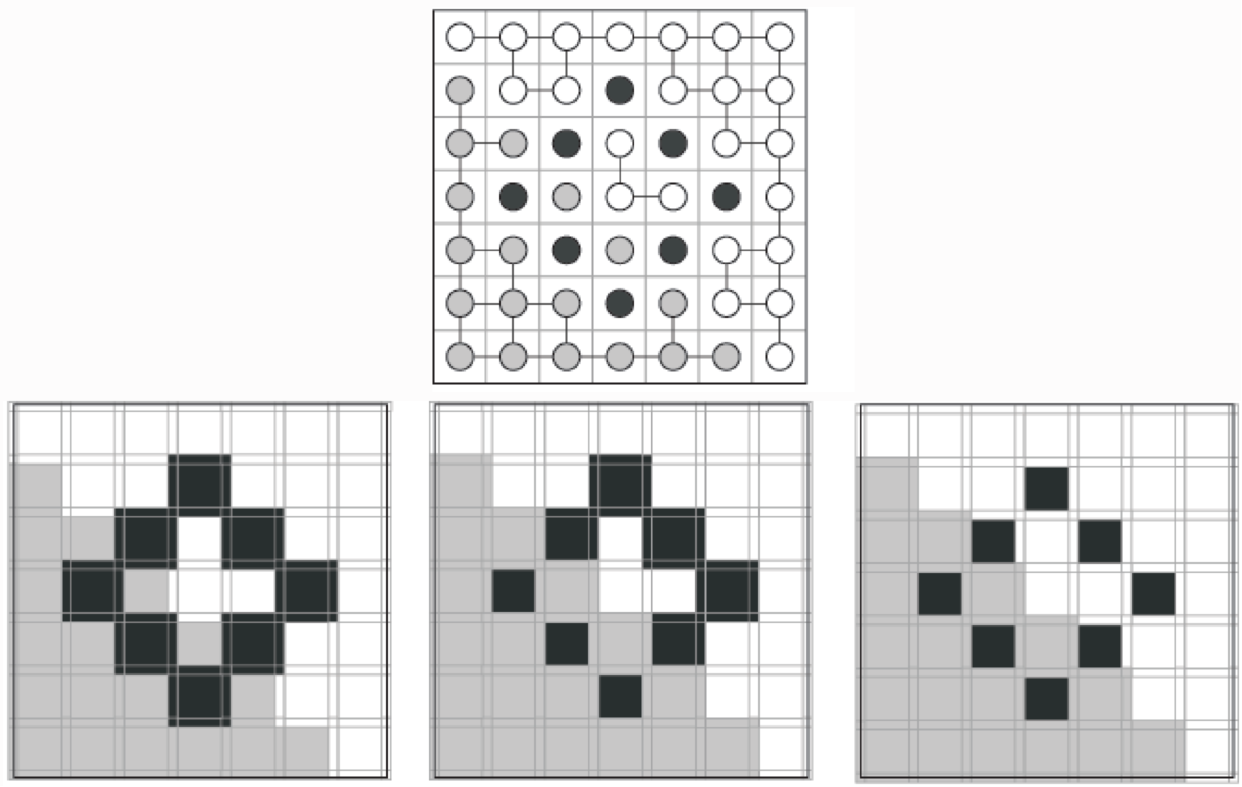
Example 1: Binary Picture



Top: (4,8) adjacency and the corresponding representation of the components in the grid cell topology.

Bottom: (8,4) adjacency.

Example 2: Three-Level Picture



Top: A three-valued input picture, showing all of the 4-components in the grid point model.

Bottom, from left to right: closed (open) regions are black (gray); closed (open) regions are gray (white); closed (open) regions are gray (black).

What are the used total orders of picture values for these three cases?

Coursework

Related material in textbook: Sections 2.1.5 and 2.2.1. (Subjects of this lecture are also related to some parts of Chapters 5 and 6, which might be scanned for additional reading.)

A.13. [5 marks] Let M be an open or closed finite set in the 2D grid cell topology, and α_m the number of m-cells in M, for $m = 0, 1, 2$.

The *Euler characteristic* of M is as follows:

$$\chi(M) = \alpha_0 - \alpha_1 + \alpha_2$$

Write a program such that for any binary picture, you can alternatively assume "white $<$ black" or "black $<$ white", and for all the resulting white or black regions calculate the Euler characteristic. (Hint: You may consider discrete integration, or you may check the literature for alternative ways of calculating $\chi(M)$.)

The complexity of the input picture should be comparable to those shown on page 12 of Lecture 08, or in the Appendix to this lecture. Discuss the obtained values of Euler characteristics with respect to the number of holes in a given region.

Appendix

Five historic American drawings from Xochicalco, Mexico (a city which existed between the years 700 and 950): specify such drawings by numbers of black components, and the area and Euler characteristics of these components. (These features might be useful for identifying similar drawings; you may use a drawing's diameter to normalize areas for scale invariance.)