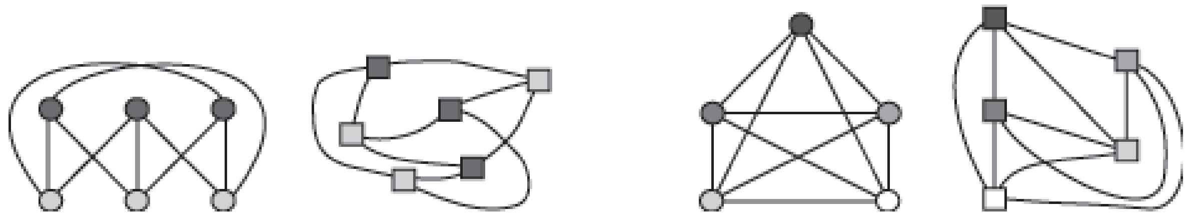
# Planarity for Graphs

$[\mathbb{Z}^2, A_8]$ shows the difficulty in defining holes for nonplanar oriented adjacency graphs (see, e.g., page 10 of Lecture 10).

informal definition: A graph $[S, A]$ is called *planar* iff it can be drawn in a plane in such a way that its edges are drawn as simple arcs that intersect only at nodes.



Two isomorphic representations of $K_{3,3}$ on the left and of $K_5$ on the right. Both are nonplanar.

Assume a finite graph. Let $\alpha_0 = \mathrm{card}(S)$ and $\alpha_1 = \mathrm{card}(A)$.

A planar drawing of a planar graph partitions the plane into $\alpha_2$ *faces* ($\alpha_2 - 1$ internal faces, and one external face).

*Euler's formula* for planar graphs:

$$\alpha_2 - \alpha_1 + \alpha_0 = 2$$

The frontier of each face defines a *cycle* $\rho$ of $\lambda(\rho)$ consecutive directed edges.. The *degree* $\nu(p)$ of node $p$ is equal to $\mathrm{card}(A(p))$.

$$\sum_{p \in S} \nu(p) = \sum_{\rho} \lambda(\rho) = 2\alpha_1$$

# Oriented Graphs

**given:** finite oriented adjacency graph $[S, A, \xi]$

**as before:** $\alpha_0 = \mathrm{card}(S)$ = number of nodes

$\alpha_1 = \mathrm{card}(A)/2$ = number of (undirected) edges

$\nu(p) = \mathrm{card}(A(p))$ = number of nodes adjacent to $p$

$\lambda(\rho)$ = length of cycle $\rho$ = number of nodes on $\rho$

**new:** $\alpha_2$ = number of cycles (cycles defined by orientation $\xi$)

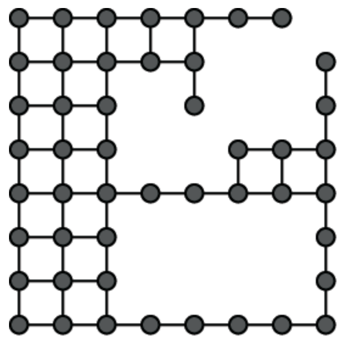
(also for nonplanar oriented graphs; if planar then $\alpha_2$ faces.)

*Euler characteristic* $\chi$ of a finite oriented adjacency graph is defined by (without a potentially infinite exterior face)

$$\chi = \alpha_0 - \alpha_1 + (\alpha_2 - 1)$$

Let $\chi^+ = \chi + 1 = \alpha_0 - \alpha_1 + \alpha_2$ (i.e., with exterior face).

**Theorem 1** $\chi^+ \leq 2$ *for any finite oriented adjacency graph.*



$\alpha_0 = 48,\ \alpha_1 = 66,\ \alpha_2 = \qquad,\ \chi^+ =$

**Definition 1** *A (finite or infinite) oriented adjacency graph is called* planar *iff either it is finite and has $\chi^+ = 2$ or is infinite and any of its nonempty finite connected oriented subgraphs has $\chi^+ = 2$.*

# Examples of Planar Oriented Graphs

1. For any planar finite graph $[S, A]$ there is an orientation $\xi$ such that $[S, A, \xi]$ is planar (i.e., with $\chi^+ = 2$).

2. $[\mathbb{Z}^2, A_4]$ using as orientation either the clockwise or counterclockwise local circular order at all of its pixels

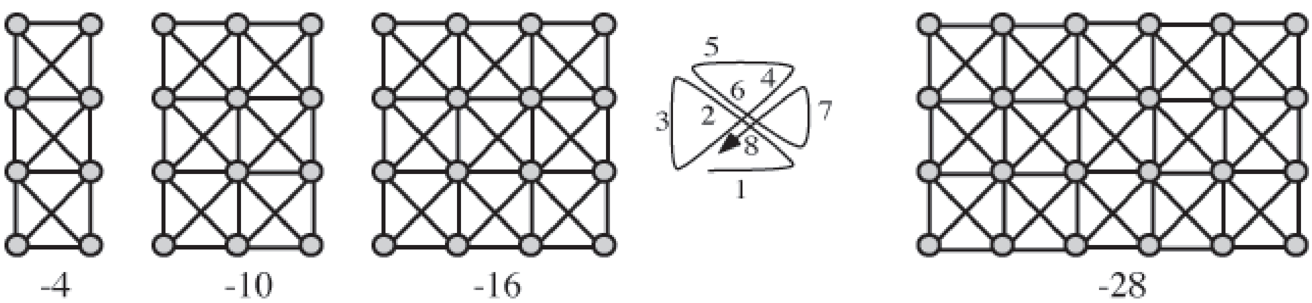
3. $[\mathbb{Z}^2, A_s]$ (i.e., switch adjacency) using as orientation either a clockwise or counterclockwise local circular order at all of its pixels

# Examples of Nonplanar Oriented Graphs

1. $K_{3,3}$ with any orientation defined on it (always $\chi^+ < 2$)

2. $K_5$ with any orientation defined on it (always $\chi^+ < 2$)

3. any subset of $[\mathbb{Z}^2, A_8]$ containing at least one $4 \times 4$ square




Unlimited decrease of the Euler characteristic $\chi^+$ in the infinite 8-adjacency grid. The numbers below the rectangular oriented graphs are values of $\chi^+$.

# The Separation Theorem

**a basic task in picture analysis:** trace the border of a region and analyze this region based on calculated properties

This requires that the performed tracing actually separates exactly all the pixels in the region from all the pixels not contained in the region, such that property calculation is based on the correct set of data. (In Euclidean geometry we have the Jordan-Veblen curve theorem which states such a separation by any simple curve.)

(K. Voss and R. Klette, 1986)

**Theorem 2** *Let $[S, A, \xi]$ be a (finite or infinite) planar oriented adjacency graph and $M$ a nonempty finite connected proper subset of $S$. Then $[S, A, \xi]$ splits into at least two nonconnected substructures when we delete the boundary of $M$.*

**Reminder:** boundary = all undirected invalid edges that are assigned to border cycles of $M$

**Conclusion:** The use of any planar oriented adjacency graph guarantees the desired separation when tracing borders.

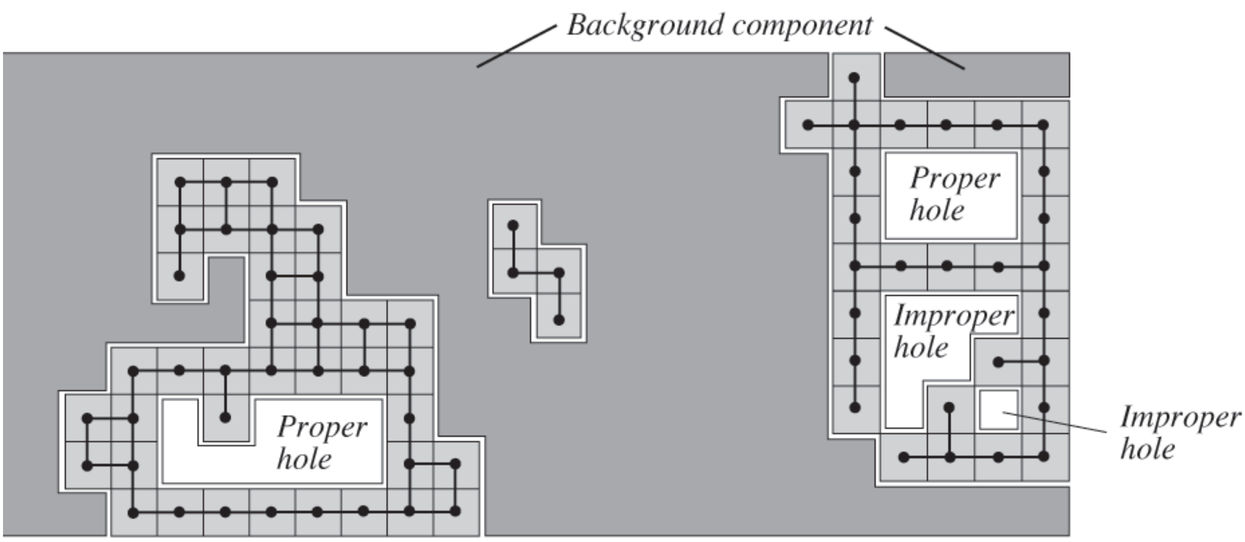
# Holes

Let $G = [S, A, \xi]$ be an infinite oriented adjacency graph and $M$ a finite connected subset of $S$.

it follows: $M$ has exactly one infinite complementary component.

**Definition 2** *Any finite complementary component of $M$ is called a* hole *of $M$.*

If $S = \mathbb{Z}^n$ and the hole is $\alpha$-connected, we call it an $\alpha$-hole. For example, regions in $[\mathbb{Z}^2, A_4]$ can have 8-holes that consist of several 4-holes.




Three 1-components and six complementary 1-components (two of these merge into one infinite background component). 1-holes in the grid cell model are 4-holes in the grid point model.

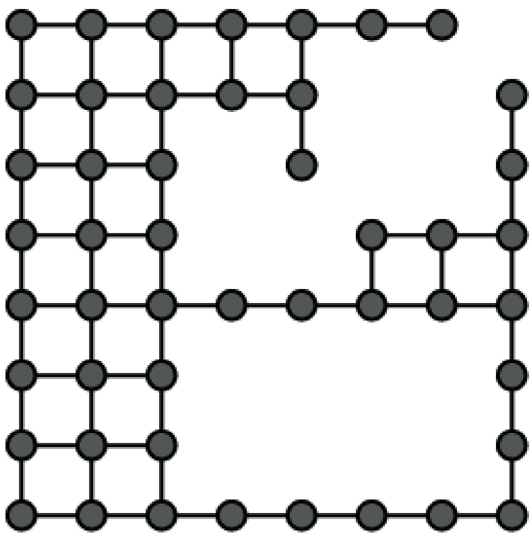
# Proper and Improper Holes

If $G$ is planar, $M$ has exactly one border cycle, called its *outer border cycle*, which separates $M$ from its infinite complementary component.

All other border cycles of $M$ are called *inner border cycles*.

If complementary component $A$ of $M$ is separated from $M$ by border cycle $\rho$ of $M$, we say that $A$ is assigned to $\rho$.

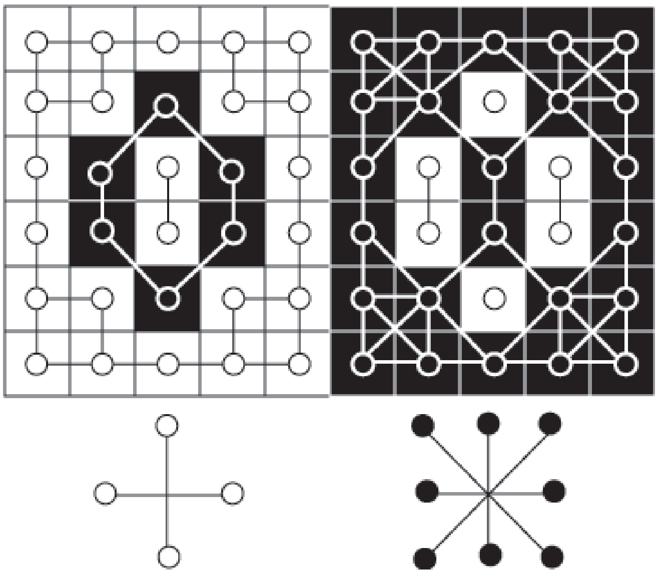
**Definition 3** *Let $M$ be a subset of an infinite planar oriented adjacency graph. A complementary component of $M$ that is assigned to one of the inner border cycles of $M$ is called a* proper hole *of $M$, and a finite complementary component that is assigned to the outer border cycle of $M$ is called an* improper hole *of $M$.*



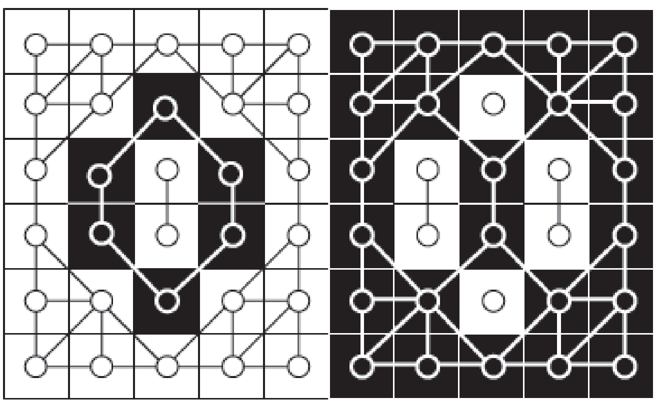
This region has one proper hole and two improper holes.

# Dual 4- and 8-Adjacency

We return to the figure shown in Lecture 02:



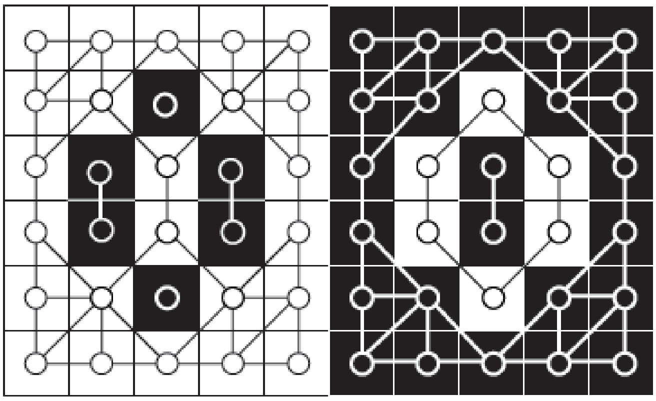
In this figure, 8-adjacency is assumed for black pixels, and 4-adjacency for white pixels (resulting into a nonplanar graph). We delete a few redundant diagonal edges between black pixels, and we add a few redundant diagonals between white pixels:



We obtain a picture in planar s-adjacency, where we prefer to draw diagonals between black pixels if there is a “flip-flop case” (i.e., two pairs of diagonally identical values).

# "White First"

If we assume 4-adjacency for black pixels, and 8-adjacency for white pixels, then we delete a few redundant diagonal edges between white pixels, and we add a few redundant diagonals between black pixels:



We obtain a picture in planar s-adjacency, where we prefer to draw diagonals between white pixels if there is a "flip-flop case" (i.e., two pairs of diagonally identical values).

# First Equivalence Theorem

**Theorem 3** *For any 2D binary picture, the dual use of 4- and 8-adjacency can be replaced by a uniform use of s-adjacency such that the resulting families of components of white or black pixels are identical for both cases.*

# Region Adjacency Graphs

The set $A(M)$ of all nodes adjacent to $M \subseteq S$ (i.e., the set of all $p \in \overline{M}$ such that $A(p) \cap M \neq \emptyset$), is called the *adjacency set* of $M$.

If $M$ is finite, so is $A(M)$.

$A(M)$ is the set of border nodes of $\overline{M}$ (= coborder of $M$).

The number of complementary components of $M$ is at most equal to the number of components of $A(M)$.

**Definition 4** *If $[S, A]$ is an adjacency graph, two disjoint subsets $M_1$ and $M_2$ of $S$ are called* adjacent *($M_1 \mathcal{A} M_2$ or $(M_1, M_2) \in \mathcal{A}$) iff $A(M_1) \cap M_2 \neq \emptyset$.*

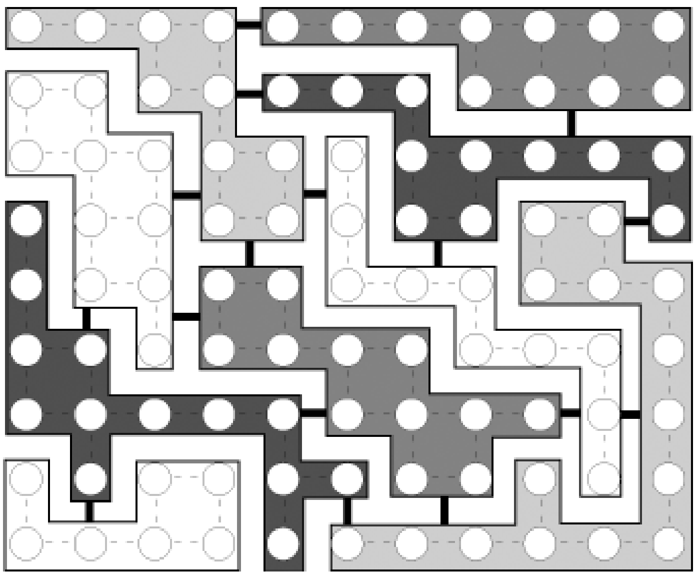
Because $A$ is symmetric, we have $A(M_1) \cap M_2 \neq \emptyset$ iff $A(M_2) \cap M_1 \neq \emptyset$ so that $\mathcal{A}$ is symmetric. Because $M_1$ and $M_2$ are disjoint, $\mathcal{A}$ is irreflexive, so it is an adjacency relation on any partition of $S$.

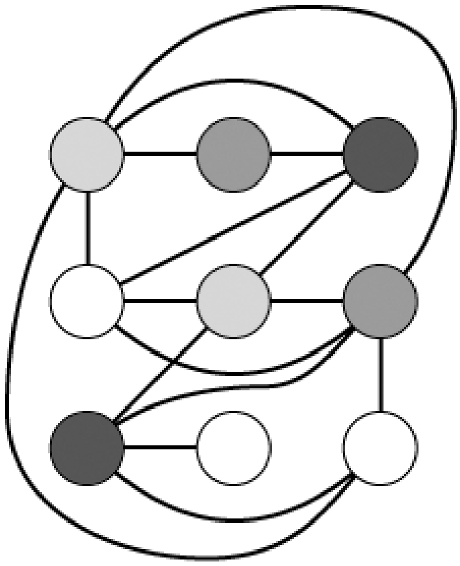
**Definition 5** *Let $\mathcal{R}$ be a partition of $S$ into regions and (possibly) the infinite background component. The undirected graph $[\mathcal{R}, \mathcal{A}]$ is the* region adjacency graph *of $\mathcal{R}$.*

Region adjacency graphs provide very useful and common ways of describing the structure of a picture.

# Example of a Multilevel Picture

We assume 4-adjacency for all pixels in the picture $P$ on the left, which already shows the resulting components of $P$-equivalence classes; edges between them illustrate region adjacency:





The graph on the right shows the region 4-adjacency graph, where each region in $P$ is now represented by just one node.

**Note:** s-adjacency has a straightforward extension to multilevel pictures. Assume a total order for picture values, and allow diagonals in flip-flop cases according to the preference defined by this total order. Discuss this for a simple example containing flip-flop cases.

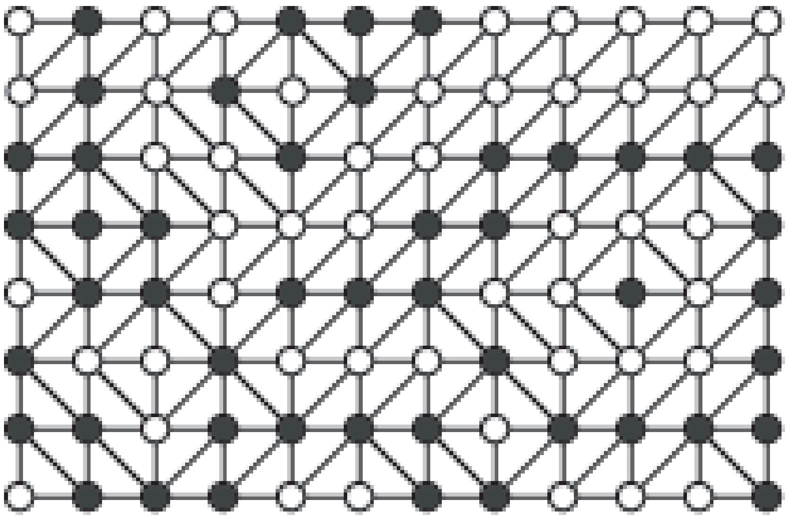
# Region s-Adjacency Graphs

A. Rosenfeld has shown in 1974:

**Theorem 4** *Let $P$ be a binary picture defined by the dual use of 4- and 8-adjacency, that is extended into $\mathbb{Z}^2$. Then the region adjacency graph of $P$ is a tree.*

**Corollary:** There are only proper holes in this case; outer border cycles of regions only separate the infinite background component from the given region (and no improper holes at all).

By applying the Equivalence Theorem (page 8) it follows that s-adjacency in binary pictures also allows to obtain region adjacency graphs in form of trees, and to exclude improper holes. Proper holes are called *holes* in these cases.



s-adjacency (uniquely defined by "black first"): draw the region adjacency graph assuming that the infinite background is white

# Coursework

Related material in textbook: Sections 4.1.4 (the final part), beginning of 4.2.1, 4.3.2, and 4.3.5. Do Exercises 9 and 11 on page 154, and on page 10 of these notes.

**A.11. [5 marks]** Implement s-adjacency for "black first" (i.e., prefer diagonals between black pixels in flip-flop cases). Your program should be able

**(i)** to interactively generate binary pictures (e.g., by drawing binary pictures of size $32 \times 32$ or larger, by cursor movement; hint: visualize them magnified on screen); show the background component by using a white frame around a generated picture,

**(ii)** to label uniquely all components of the picture (hint: FILL algorithm; will be provided), and

**(iii)** (in a final interactive step) after moving the cursor onto a component $A$, all the components adjacent to $A$ should be highlighted (hint: by increasing intensity values; this is another use of FILL) and

**(iv)** below the curser (or in another window) list all labels of those components which define holes in $A$.