

**CDMTCS
Research
Report
Series**

**Unstable Dynamics on a
Markov Background and
Stability in Average**

Alexander M. Krägeloh
Department of Mathematics
University of Auckland

CDMTCS-097
March 1999

Centre for Discrete Mathematics and
Theoretical Computer Science

UNSTABLE DYNAMICS ON A MARKOV BACKGROUND
AND STABILITY IN AVERAGE

Alexander Marc Krägeloh

Abstract

Dynamical systems which switch between several different branches or modes of evolution via a Markov process are simple mathematical models for irreversible systems. The averaged evolution for such a dynamics can be obtained by a compression of the corresponding reversible dynamics onto a coinvariant subspace in the sense of the Lax/Phillips Scattering Scheme.

If the dynamics switches between some stable modes of evolution and some unstable modes, still the averaged or expectation evolution might be stable. From the theory of random evolutions the generator \mathcal{A} of the averaged evolution is obtained, and a definition of stability in average is suggested. With regard to this context the generator \mathcal{A} is investigated and conditions for stability in average are given for certain special situations. Based on these results, a conjecture is made about sufficient and necessary conditions for stability in average for some more general cases.

In order to find hints how to verify or how to modify the conjecture three qualitatively different solvable models are studied. Here the spectral properties of the generators of all three models are studied, and the results are put in relation to the conjecture.

Preface

This booklet is a revised and extended version of my Master thesis in Mathematics which I had completed in February 1998 at The University of Auckland, New Zealand, under the supervision of Professor Boris Pavlov.

It is revised not only in the way that I have modified the layout, reduced the number of misprints and changed the formulations here and there, but I have also removed the detailed codes of the programmes used, and shortened some of the calculations in the appendices. More important, however, is the modification of the definition of 'stability in average' (see Section 2.3).

Professor Boris Pavlov and I have continued our discussions on the topic and its problems since the thesis was submitted. The above mentioned extension consists of our results on the properties of the operators studied, now we are able to formulate our conjecture in more general terms and base it on some rigorous result on the dissipativity of the generator (see Section 2.3). In August 1998 we submitted these results as a joint paper to a journal.

I am grateful to Professor Boris Pavlov for his continued guidance and suggestions in this field and for his advice to this text, and I acknowledge the valuable comments from Professor Vadim Adamjan, especially towards the definition of 'stability in average'.

I wish to thank Professor Ioannis Antoniou and the Solvay Institutes, Brussels, for the hospitality and the inspiring discussions during the conference on the EC-project ESPRIT in June 1998. In Brussels Professor Cristian Calude suggested the printing of this text in this series, I wish to thank him for his interest.

The financial support from the Department of Mathematics, the Faculty of Science and the Scholarships Office of The University of Auckland during the last year of my MSc course is gratefully acknowledged. The subsequent results were received within the programme of Professor Boris Pavlov's Marsden Fund Grant 3368152, I also wish to recognise this support.

March 1999

Alexander M. Krägeloh
Universität der Bundeswehr München
Fakultät für Informatik
D-85577 Neubiberg, Germany

Contents

1	Introduction	1
2	Random evolution and Stability in Average	4
2.1	Preliminaries about stochastic processes	4
2.1.1	The Poisson process	4
2.1.2	The continuous-time n -state Markov chain	5
2.2	Random evolution	10
2.2.1	Forward and backward evolution	10
2.2.2	The expectation semi-group	12
2.3	Stability in Average	17
2.4	The generator \mathcal{A} of the expectation semi-group for a 2-state Markov chain	24
2.4.1	The Frobenius-Schur factorisation of \mathcal{A}	24
2.4.2	\mathcal{A} for small intensity	25
2.4.3	The adjoint operator of \mathcal{A}	26
2.4.4	The resolvent of \mathcal{A}	26
3	A point mass in a random potential	28
3.1	Description of the model	28
3.1.1	The stable dynamics	30
3.1.2	The unstable dynamics	31
3.1.3	The expectation semi-group	31
3.2	Conditions for stability in average	32
3.2.1	Stable potential 'weaker' than unstable potential	32
3.2.2	Stable potential 'stronger' than unstable potential	33
3.3	Summary and graphical illustration	37
4	The wave equation on the finite string with random boundary conditions	39
4.1	Description of the model	40
4.1.1	The case of a general density function	41
4.1.2	A solvable model	45
4.1.3	The stable system	48
4.1.4	The unstable system	50

4.2	Some relations between the generators of the stable and the unstable dynamics	57
4.2.1	The domains	57
4.2.2	The energy forms	57
4.2.3	The eigenvalues and eigenvectors	58
4.2.4	The eigenvector expansions	58
4.2.5	The difference of the resolvents	61
4.3	The generator \mathcal{A} of the expectation semi-group	62
4.3.1	Asymptotic behaviour of the spectrum of \mathcal{A}	63
4.3.2	Perturbation Theory and Stability in Average	64
4.3.3	The resolvent of the generator \mathcal{A}	65
4.4	Numerical approximation of the spectrum of \mathcal{A}	66
4.4.1	Preliminaries about the Galerkin method	66
4.4.2	The Galerkin method applied to the generator \mathcal{A}	67
4.4.3	Results of the numerical approximation	69
4.5	Summary	71
5	The wave equation on the semi-infinite string divided by a point mass and with random boundary conditions	73
5.1	The Lax/Phillips scattering scheme applied to the wave equation on the semi-infinite string	74
5.2	Description of the problem - a solvable model	77
5.2.1	The point mass and the differential operator L	78
5.2.2	The reflection coefficient and the negative eigenvalue of the differential operator L	79
5.2.3	The stable system	84
5.2.4	The unstable system	85
5.2.5	The expectation semi-group	86
5.3	The energy approach to stability	86
6	Summary and further developments	89
A	Theory of Linear Operators and Inner Product Spaces	91
A.1	Linear operators on Hilbert space	91
A.2	One-parameter semi-groups of operators	95
A.3	Perturbation Theory	98
A.4	A differential operator on $\mathcal{L}_2[0, N]$	99
A.5	Linear spaces with an indefinite metric	104
A.6	Hardy spaces	106
B	Attachments to Chapter 3	109
C	Attachments to Chapter 4	114
C.1	General calculations	114

C.1.1	The closure of the domain of A	114
C.1.2	The energy norms of the eigenvectors U_s, V_s and Ψ_{\pm}	115
C.1.3	The restriction of A_2 onto Π_-	117
C.1.4	The adjoint operators of A_0	117
C.1.5	Estimation of $\ U_{s+1} \Leftrightarrow V_s\ _j^2, j = 1, 2$	118
C.2	The Galerkin method	120
C.2.1	The system matrix ,	120
C.2.2	Estimation of Δ_K	121
C.2.3	The programme codes	126
C.2.4	Some graphs	128
References		133
Abbreviations and Notations		136

Chapter 1

Introduction

In the paper “Quantum dynamics on a Markov background“ by B.S. Pavlov ([28]) a certain class of dynamical systems was constructed which has the property of being irreversible. It was assumed that the generator of the evolution depends on time via a continuous time n -state Markov process, it jumps between the self-adjoint operators A_1, \dots, A_n , defined on a Hilbert space H .

From this *random evolution* $U(t)$, which is not a semi-group, the *averaged evolution* $Z(t)$ was taken (see e.g. Chermshantsev[5] or Pinsky[30]). This evolution is a semi-group, also called *expectation semi-group*, and it is generated by a dissipative operator L . In fact it was shown that a self-adjoint dilation \hat{L} on a Hilbert space $\hat{H} \supset H$ can be constructed, so that the unitary group $e^{i\hat{L}t}$ has incoming and outgoing subspaces \mathcal{D}_- and \mathcal{D}_+ in the sense of the Lax/Phillips scattering scheme, and such that the averaged evolution $Z(t) = e^{iLt}$ is the compression $P_K U(t)|_K$ onto the coinvariant subspace $K = \hat{H} \ominus (\mathcal{D}_+ \oplus \mathcal{D}_-)$. This dilation plays the role of the underlying reversible dynamics, so that it can be studied together with the irreversible compression of it onto the corresponding coinvariant subspace.

In this text we are interested in the averaged stability of the class of dynamical systems affected by a 2-state Markov process with ‘intensity’ \varkappa , where the evolution ‘jumps’ between a stable and an unstable mode. To be more specific, the generator A_1 of the stable branch of evolution shall be self-adjoint or dissipative, and the generator of the unstable branch of evolution is assumed to have a one-dimensional accretive part, and on its complementary subspace it is self-adjoint or dissipative. If the averaged evolution for some choice of the parameter \varkappa of the Markov chain is a semi-group of uniformly bounded operators then the system is called **stable in average**. We investigate the necessary and/or sufficient conditions for stability in average. For a special case some answer to this question is given, for more general cases a conjecture is made (cf. Conjecture 2.34) which can be formulated to suit above situation as follows

Conjecture 1.1

Let $\pi = (\pi_1, \pi_2)$ be the equilibrium distribution of the Markov chain. Then a sufficient and necessary condition for possible stability in average is that the operator $\hat{A} = \pi_1^2 A_1 + \pi_2^2 A_2$ generates a uniformly bounded semi-group, that means that there exists a constant $K > 0$ such that $\|e^{i\hat{A}t}\| \leq K$ for all $t \geq 0$.

In order to find hints how to verify or how to modify this conjecture we shall investigate the question of stability on three solvable models with the characteristics:

- (i) both A_1 and A_2 are bounded operators;
- (ii) both A_1 and A_2 are unbounded, A_1 is self-adjoint, and A_2 has a one-dimensional accretive part and is self-adjoint on the complementary subspace;
- (iii) both A_1 and A_2 are unbounded, A_1 is dissipative, and A_2 has a one-dimensional accretive part, but is dissipative on the complement.

A model for the situation (i) is studied in Chapter 3, a point mass in a random potential. The wave equation on the finite string with random boundary conditions is studied in Chapter 4 as an example of situation (ii). Then, in Chapter 5 we investigate the wave equation on the semi-infinite string divided by a point mass and with random boundary conditions. The evolution which interests us here is the evolution on the finite part of the string separated by the point mass. Here we apply the Lax/Phillips scattering scheme to obtain the dissipative operators.

RANDOM EVOLUTION

Motivated by a connection found by Mark Kac (see “Some stochastic problems in physics and mathematics”, *Magnolia Petroleum Co, Lectures in Pure and Applied Science*, no 2 (1956)) between a simple random movement of a point mass and the telegraph equation

$$\frac{1}{c} \frac{\partial^2}{\partial t^2} u = c \frac{\partial^2}{\partial x^2} u \Leftrightarrow \frac{2a}{c} \frac{\partial}{\partial t} u,$$

R. Griego and R. Hersh introduced the notion of random evolution in 1969 (Griego/Hersh[9]). It is connected with the Cauchy problem

$$\begin{cases} \frac{1}{i} \frac{d}{dt} y(t) = A_{\nu(t)} y(t) \\ y(0) = y_0 \end{cases}$$

where $\nu(t)$ is some random process (e.g. Markov chain), and to each $\nu(t)$ corresponds a generator of a C_0 -semi-group of operators, exactly what was constructed in Pavlov[28]. The original intention was not to actually study systems in which the mode of evolution changes randomly in time, but to solve certain other Cauchy problems and find limit theorems in connection with a general telegraph equation like

$$u_{tt} + 2au_t = A^2 u.$$

An overview of the results and problems for the time up to 1974 concerning random evolutions is given in a survey by R. Hersh ([13]). Only later were actual systems with random evolution investigated. S.E. Cheremshantsev ([5]) studied the scattering problem on the Schrödinger equation

$$i \frac{\partial}{\partial t} \psi = \Leftrightarrow \Delta_x \psi + q(x \Leftrightarrow y(t)) \psi,$$

where the potential $q(x \Leftrightarrow y(t))$ depends on time via a Brownian motion $y(t)$. The averaged dynamics was constructed and the mean scattering operator was computed.

In the last chapter of a book by M. Pinsky ([30]) the question of stability of random evolutions is treated. However, this is done only for the case that some small stochastic noise is superposed to an oscillation, not for the case that the mode of oscillation changes drastically. A long list of references of work done in this field up to 1991 is given in this book.

In Chapter 2 of this text are listed the definitions of Poisson processes and Markov chains, and also the definitions of the random evolution and the expectation semi-group. The generator of the semi-group is the operator considered in Pavlov[28]. Some properties of this operator are found as well as some conditions for stability in average. The spectral properties of the generator for all three solvable models (i), (ii), and (iii) with regard to the question of stability are studied in the Chapters 3, 4, and 5. Then we relate any result to the Conjecture 2.34 in Chapter 6.

Appendix A contains some definitions and propositions from the Theory of Linear Operators, Semi-groups of Operators, Perturbation Theory, Theory of Differential Operators, Krein spaces, and Hardy spaces, which will be quoted in the text. Longer calculations, graphs and some details on programmes run to produce these graphs are collected as attachments to the Chapters 3 and 4 in the Appendices B and C respectively.

Chapter 2

Random evolution and Stability in Average

The notion of *random evolution* and *expectation semi-group* was introduced by R.Griego and R.Hersh ([9]) in 1969. It provides a tool to study the question of averaged stability of dynamics on a Markov background given by a Cauchy problem.

In the first Section 2.1 are listed the definitions of Poisson processes and Markov chains, together with some properties. These are used in Section 2.2 to give the definitions of random evolution and expectation semi-group. Some properties are then quoted from the respective literature and proofs given in detail.

A definition for stability in average is suggested in Section 2.3, we study the generator of the expectation semi-group and find some conditions for averaged stability.

Then, in Section 2.4, some observations are made about the infinitesimal generator of the expectation semi-group for the special case of a continuous-time and symmetric 2-state Markov chain.

2.1 Preliminaries about stochastic processes

The definitions in this section can be found in any standard book on stochastic processes, for instance in Breiman[3], Grimmett/Stirzaker[11] or Taylor/Karlin[34].

2.1.1 The Poisson process

Definition 2.1

Let X be a random variable with values in the positive reals \mathbb{R}^+ .

X is said to have **exponential distribution** with **intensity** \varkappa if $\Pr(X > t) = e^{-\varkappa t}$.

The density function such a random variable X is

$$\Pr(X \in dr) = \varkappa e^{-\varkappa r} dr. \tag{2.1}$$

An important property of the exponential distribution is the so called **memory-less property**: If we think of X as the life-time of a certain unit and assume that the unit has survived up to time $t > 0$, then the random variable $X \ominus t$ of the remaining life-time under the condition that it has survived up to time t is also exponentially distributed

with intensity \varkappa , since for $s > 0$

$$\Pr(X \Leftrightarrow t > s | X > t) = \frac{\Pr(X \Leftrightarrow t > s)}{\Pr(X > t)} = \frac{e^{-\varkappa(s+t)}}{e^{-\varkappa t}} = e^{-\varkappa s}.$$

Definition 2.2

A family $\mathcal{N} = \{N(t) : t \geq 0\}$ of random variables with values in the state space $S = \mathbb{N}_0$ and with $N(t) \leq N(t+s)$ for all $s, t \geq 0$ is called a **Poisson process** with **intensity** \varkappa if the properties (P1)-(P3) hold:

(P1) for any set of times $t_0 = 0 < t_1 < \dots < t_n$ the increments $N(t_1) \Leftrightarrow N(t_0), \dots, N(t_n) \Leftrightarrow N(t_{n-1})$ are independent random variables,

(P2) for $s, t > 0$ the random variable $N(t+s) \Leftrightarrow N(s)$ has Poisson distribution with mean $\varkappa t$, i.e.

$$\Pr(N(t+s) \Leftrightarrow N(s) = k) = \frac{(\varkappa t)^k e^{-\varkappa t}}{k!}, \quad k \in \mathbb{N}_0$$

(P3) $N(0) = 0$.

2.1.2 The continuous-time n -state Markov chain

Definition 2.3

A family $\nu = \{\nu(t) : t \geq 0\}$ of random variables with values in a finite state space $S = \{1, \dots, n\}$ is called a **continuous-time n -state Markov chain** or **process** if it satisfies the **Markov property**:

For all finite sets of times $0 \leq t_1 < \dots < t_n < t$ and states $k, j_1, \dots, j_n \in S$ it is

$$\Pr(\nu(t) = k | \nu(t_1) = j_1 \wedge \dots \wedge \nu(t_n) = j_n) = \Pr(\nu(t) = k | \nu(t_n) = j_n). \quad (2.2)$$

The Markov property (2.2) is equivalent to the condition that for all $k, j, l(u) \in S$ ($0 \leq u \leq s$)

$$\Pr(\nu(s+t) = k | \nu(s) = j \wedge \nu(u) = l(u), 0 \leq u < s) = \Pr(\nu(s+t) = k | \nu(s) = j). \quad (2.3)$$

TRANSITION PROBABILITIES AND THE INFINITESIMAL GENERATOR

Definition 2.4

(1) If the **transition probabilities** $\Pr(\nu(s+t) = j | \nu(s) = k)$ are independent of $s \geq 0$ then the Markov chain is said to have **stationary** or **homogeneous** transition probabilities. In this case we define

$$p_{jk}(t) := \Pr(\nu(t) = k | \nu(0) = j).$$

(2) The Markov chain is said to be **regular** if

$$\lim_{t \rightarrow 0} p_{jk}(t) = \delta_{jk} \quad \forall j, k \in S.$$

(3) The **probability distribution** at time t of the Markov chain is the vector

$$p(t) := (p_1(t), \dots, p_n(t)) := (\Pr(\nu(t) = 1), \dots, \Pr(\nu(t) = n)),$$

$p(0)$ is called the **initial distribution**.

(4) The set of **jump times** $(\tau_n)_{n \in \mathbb{N}}$ is defined by

$$\tau_0 = 0, \quad \tau_{n+1} = \inf\{t > \tau_n : \nu(t) \neq \nu(\tau_n)\}, \quad n \in \mathbb{N}.$$

(5) The **jump probabilities** $(J_{jk})_{j,k \in S}$ are

$$J_{jk} := \Pr(\nu(\tau_1) = k | \nu(0) = j).$$

Certainly it is

$$\sum_{k=1, k \neq j}^n J_{jk} = 1, \quad j \in S. \quad (2.4)$$

The set $\{p_{jk}(t) : j, k \in S\}$ of transition probabilities for a continuous-time, regular n -state Markov chain also satisfies the conditions (see e.g. Breiman[3]):

$$(i) \quad \sum_{k=1}^n p_{jk}(t) = 1 \quad j \in S$$

$$(iii) \quad p_{jk}(t+s) = \sum_{l=1}^n p_{jl}(t)p_{lk}(s)$$

(Chapman-Kolmogorov forward equation)

Definition 2.5

The **transition matrix** $P(t)$ is defined as

$$P(t) = (p_{jk}(t))_{j,k=1}^n = \begin{pmatrix} p_{11}(t) & \cdots & p_{1n}(t) \\ \vdots & \ddots & \vdots \\ p_{n1}(t) & \cdots & p_{nn}(t) \end{pmatrix}.$$

The left-eigenvector π of $P(t)$ to the eigenvalue 1 is called the **equilibrium distribution**, i.e. it is $\pi P(t) = \pi$.

With the help of the transition matrix $P(t)$ the distribution $p(t)$ at time $t \geq 0$ can easily be calculated from the initial distribution $p(0)$. We have

$$\begin{aligned} p_j(t) &= \Pr(\nu(t) = j) \\ &= \sum_{l=1}^n \Pr(\nu(0) = l) \Pr(\nu(t) = j | \nu(0) = l) = \sum_{l=1}^n p_l(0) p_{lj}(t) \end{aligned}$$

and hence

$$p(t) = p(0)P(t).$$

It is easily verified that the family $\{P(t) : t \geq 0\}$ has in fact the properties (SG1)-(SG3) of a one-parameter semi-group of operators listed in Appendix A.2.

Definition 2.6

For the Markov chain with (stochastic) semi-group $\{P(t) : t \geq 0\}$ define the **infinitesimal generator** Q

$$Q := \lim_{t \rightarrow 0} \frac{P(t) \Leftrightarrow I}{t}$$

Then Q and $P(t)$ satisfy

$$\frac{d}{dt}P(t) = QP(t) = P(t)Q$$

For the row sums of Q we get

$$\sum_{k=1}^n q_{jk} = \lim_{t \rightarrow 0} \sum_{k=1}^n \frac{p_{jk}(t) \Leftrightarrow p_{jk}(0)}{t} = \lim_{t \rightarrow 0} \frac{1}{t} \left(\sum_{k=1}^n p_{jk}(t) \Leftrightarrow 1 \right) = 0. \quad (2.5)$$

In Grimmett[11] it is shown that from the Markov property and homogeneity follows that the sojourn times τ are exponentially distributed with some intensity q_j dependent on the state j the chain is in. In fact one has for $j \in S$ with some $q_j > 0$

$$\Pr(\tau > t | \nu(0) = j) = e^{-q_j t}$$

and with the jump probabilities one gets

$$\Pr(\nu(\tau) = k, \tau > t | \nu(0) = j) = J_{jk} e^{-q_j t}. \quad (2.6)$$

It follows for $j \neq k$

$$q_{jk} = \left. \frac{d}{dt} p_{jk}(t) \right|_{t=0} = q_j J_{jk},$$

and with (2.4) and (2.5)

$$\Leftrightarrow q_{jj} = \sum_{k=1, k \neq j}^n q_{jk} = q_j,$$

that is

$$q_j = \Leftrightarrow q_j \quad \text{and} \quad J_{jk} = \Leftrightarrow \frac{q_{jk}}{q_{jj}} \quad \text{for } j = 1, \dots, n \quad (2.7)$$

Definition 2.7

Define the random variable $N(t)$ which counts the number of jumps up to (and including) time t . And let $N_s(t)$ be the number of jumps in the interval $(s, t]$.

The following proposition, also to be found in Griego/Hersh[10], gives an estimate of the probabilities for m jumps of the chain up to time t .

Proposition 2.8

For $m \in \mathbb{N}_0$ we have

$$\exists c_1, c_2 > 0 : \Pr(N(t) = m) \leq \frac{(c_1 t)^m}{m!} e^{-c_2 t} \quad (2.8)$$

Proof: (by induction)

Define $c_1 := \max\{q_k; k = 1, \dots, n\} > 0$ and $c_2 := \min\{q_k; k = 1, \dots, n\} > 0$.

Let $m = 0$, then

$$\begin{aligned}\Pr(N(t) = 0) &= \sum_{k=1}^n \Pr(N(t) = 0 | \nu(0) = k) \Pr(\nu(0) = k) \\ &= \sum_{k=1}^n e^{-q_k t} p_k(0) \leq e^{-c_2 t}.\end{aligned}$$

Assume the assertion holds for $m \Leftrightarrow 1$, then we get for m (with conditioning on the time τ of the first jump):

$$\begin{aligned}\Pr(N(t) = m) &= \int_0^t \Pr(N(t) = m | \tau = r) \Pr(\tau \in dr) \\ &= \int_0^t \Pr(N_r(t \Leftrightarrow r) = m \Leftrightarrow 1) \sum_{k=1}^n \Pr_k(\tau \in dr) \\ &\leq \int_0^t \frac{[c_1(t \Leftrightarrow r)]^{m-1}}{(m \Leftrightarrow 1)!} e^{-c_2(t-r)} \sum_{k=1}^n q_k e^{-q_k r} dr \\ &\leq \frac{c_1^m}{(m \Leftrightarrow 1)!} e^{-c_2 t} \int_0^t (t \Leftrightarrow r)^{m-1} dr = \frac{(c_1 t)^m}{m!} e^{-c_2 t}\end{aligned}$$

□

TRAJECTORIES

When we follow a realisation of a continuous-time Markov chain we note the state $\nu(t)$ in which it is at each point of time t and the jump times $(\tau_j)_{j \in \mathbb{N}}$ at which the states change. The function $\nu(t)$ is a right-continuous piecewise constant function with values in $S = \{1, \dots, n\}$. This gives rise to the

Definition 2.9

- (1) A (infinite) collection of pairs

$$\omega = \{(0, \sigma_0), (\tau_1, \sigma_1), \dots, (\tau_m, \sigma_m), \dots\},$$

where τ_j are jump times and σ_j are the states in which the Markov chain ν is in the time interval $[\tau_j, \tau_{j+1})$ is called a **trajectory** or **sample path**. Let $\omega(t)$ be the corresponding piecewise constant function. From the definition of $N(t)$ we get $\omega(t) = \sigma_{N(t)}$.

- (2) With the **shifted trajectory** by time $s > 0$ we mean the trajectory

$$\tilde{\omega}_s = \{(0, \tilde{\sigma}_0), (\tilde{\tau}_1, \tilde{\sigma}_1), \dots, (\tilde{\tau}_m, \tilde{\sigma}_m), \dots\}$$

where

$$\begin{aligned}\tilde{\tau}_j &= \tau_{N(s)+j} \Leftrightarrow s, & j &\geq 1 \\ \tilde{\sigma}_j &= \sigma_{N(s)+j}, & j &\geq 0\end{aligned}\tag{2.9}$$

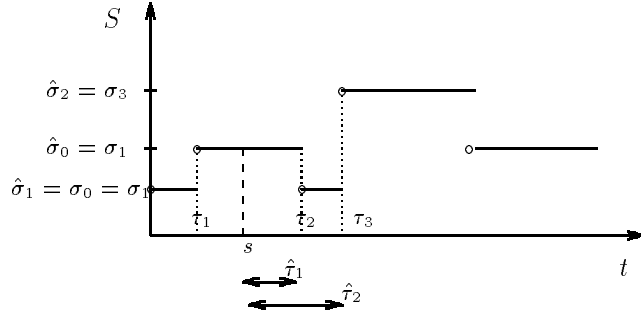


Figure 2.1: The shifted trajectory

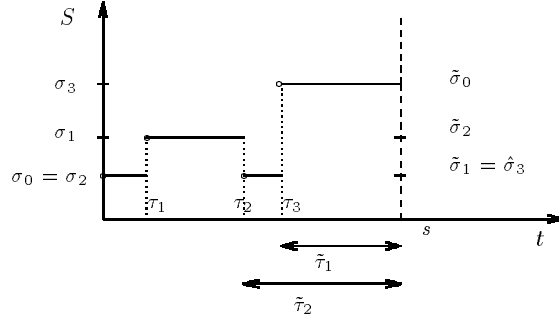


Figure 2.2: The reverse trajectory

(3) A **finite trajectory** up to time s (of **length** s) we denote with

$$\omega[s] = \{(0, \sigma_0), (\tau_1, \sigma_1), \dots, (\tau_{N(s)}, \sigma_{N(s)})\}$$

(4) For a finite trajectory we can consider the **reverse trajectory**

$$\hat{\omega}[s] = \{(0, \hat{\sigma}_0), (\hat{\tau}_1, \hat{\sigma}_1), \dots, (\hat{\tau}_{N(s)}, \hat{\sigma}_{N(s)})\}$$

with

$$\begin{aligned} \hat{\tau}_j &= s \Leftrightarrow \tau_{N(s)} & , \quad j = 1, \dots, N(s) \\ \hat{\sigma}_j &= \sigma_{N(s)-j} & , \quad j = 0, \dots, N(s) \end{aligned} \tag{2.10}$$

From above definitions and the Markov property follows

Lemma 2.10

(1) Let ω be a trajectory and $t \geq 0$, then

$$\Pr(\omega | \omega[t]) = \Pr(\tilde{\omega}_t) \tag{2.11}$$

(2) Let $P(t)$ be symmetric and $s \geq 0$, then

$$\Pr(\omega[s]) = \Pr(\hat{\omega}[s])$$

2.2 Random evolution

Let A_1, \dots, A_n be infinitesimal generators of C_0 -semi-groups T_1, \dots, T_n on the Hilbert spaces $H_1 = (H, [\cdot, \cdot]_1), \dots, H_n = (H, [\cdot, \cdot]_n)$, and let $M \geq 1$ and $\beta \geq 0$ be such that

$$\|T_j(t)\|_j \leq M e^{\beta t}, \text{ for } j = 1, \dots, n. \quad (2.12)$$

The domains $\mathcal{D}(A_j) \subset H, j = 1, \dots, n$, need not be the same subspaces of the vector space H , but the inner products are assumed to be at least mutually equivalent. We define the two sets of positive constants $(\gamma_{jk})_{j,k=1}^n$ and $(,_{jk})_{j,k=1}^n$ by

$$\gamma_{jk} \|\cdot\|_k \leq \|\cdot\|_j \leq ,_{jk} \|\cdot\|_k \quad , j, k \in S.$$

Further let $\nu = \{\nu(t) : t \geq 0\}$ be a continuous-time Markov chain on the finite state space $S = \{1, \dots, n\}$ with infinitesimal generator $Q = (q_{jk})_{j,k=1}^n$ and transition matrix $P(t) = e^{Qt}$. The random evolution is connected with the Cauchy problem

$$\begin{cases} \frac{1}{i} \frac{d}{dt} y(t) = A_{\nu(t)} y(t) \\ y(0) = y_0 \end{cases} \quad (2.13)$$

2.2.1 Forward and backward evolution

Definition 2.11

For a given trajectory $\omega = \{(0, \sigma_0), (\tau_1, \sigma_1), \dots, (\tau_m, \sigma_m), \dots\}$ the following operators are defined:

- (1) the **backward random evolution** (Griego/Hersh[9] and [10])

$$M_b(t, \omega[t]) := T_{\sigma_0}(\tau_1) T_{\sigma_1}(\tau_2 \Leftrightarrow \tau_1) \cdots T_{\sigma_{N(t)}}(t \Leftrightarrow \tau_{N(t)}),$$

- (2) and the **forward random evolution** (Keeper[18] and [20])

$$M_f(t, \omega[t]) := T_{\sigma_{N(t)}}(t \Leftrightarrow \tau_{N(t)}) \cdots T_{\sigma_1}(\tau_2 \Leftrightarrow \tau_1) T_{\sigma_0}(\tau_1).$$

Obviously, if the operators $T_j, j = 1, \dots, n$, commute, the backward and forward random evolutions coincide.

Immediately from the definitions of the random evolutions and the reverse and shifted trajectories as well as (2.12) follows

Lemma 2.12

- (1) For a trajectory ω with $N(t) = m$ and $k \in S$ it is

$$\begin{aligned} \|M_b(t, \omega[t])\|_k &\leq (\max_j |,_{kj}|)^{m+1} M^{m+1} e^{\beta t} \\ \|M_f(t, \omega[t])\|_k &\leq (\max_j |,_{kj}|)^{m+1} M^{m+1} e^{\beta t} \end{aligned}$$

- (2) $M_b(t, \omega[t]) = M_f(t, \hat{\omega}[t])$

$$(3) \quad M_b(t, \omega[t]) = T_{\omega(0)}(\tau_1)M_b(t \Leftrightarrow \tau_1, \tilde{\omega}_{\tau_1}[t \Leftrightarrow \tau_1])$$

$$M_f(t, \omega[t]) = M_f(t \Leftrightarrow \tau_1, \tilde{\omega}_{\tau_1}[t \Leftrightarrow \tau_1])T_{\omega(0)}(\tau_1)$$

Proposition 2.13

If $t \neq \tau_k$ for $k \in \mathbb{N}$, then

$$(1) \quad \frac{1}{i} \frac{d}{dt} M_b(t, \omega[t]) = M_b(t, \omega[t]) A_{\sigma_{N(t)}}$$

$$(2) \quad \frac{1}{i} \frac{d}{dt} M_f(t, \omega[t]) = A_{\sigma_{N(t)}} M_f(t, \omega[t])$$

Proof:

(1) For $j = 1, \dots, n$ is A_j the generator of the semi-group T_j , thus

$$\frac{1}{i} \frac{d}{dt} T_j(t) = A_j T_j = T_j A_j$$

from Theorem A.23. $N(t)$ is constant for $t \neq \tau_k, k \in \mathbb{N}$, therefore

$$\begin{aligned} \frac{1}{i} \frac{d}{dt} M_b(t, \omega[t]) &= T_{\sigma_0}(\tau_1) \cdots T_{\sigma_{N(t)-1}}(\tau_{N(t)} \Leftrightarrow \tau_{N(t)-1}) \frac{1}{i} \frac{d}{dt} T_{\sigma_{N(t)}}(t \Leftrightarrow \tau_{N(t)}) \\ &= T_{\sigma_0}(\tau_1) \cdots T_{\sigma_{N(t)-1}}(\tau_{N(t)} \Leftrightarrow \tau_{N(t)-1}) T_{\sigma_{N(t)}}(t \Leftrightarrow \tau_{N(t)}) A_{\sigma_{N(t)}} \\ &= M_b(t, \omega[t]) A_{\sigma_{N(t)}}. \end{aligned}$$

(2) and similarly

$$\begin{aligned} \frac{1}{i} \frac{d}{dt} M_f(t, \omega[t]) &= \frac{1}{i} \frac{d}{dt} T_{\sigma_{N(t)}}(t \Leftrightarrow \tau_{N(t)}) \cdots T_{\sigma_1}(\tau_2 \Leftrightarrow \tau_1) T_{\sigma_0}(\tau_1) \\ &= A_{\sigma_{N(t)}} T_{\sigma_{N(t)}}(t \Leftrightarrow \tau_{N(t)}) \cdots T_{\sigma_1}(\tau_2 \Leftrightarrow \tau_1) T_{\sigma_0}(\tau_1) \\ &= A_{\sigma_{N(t)}} M_f(t, \omega[t]). \end{aligned}$$

□

Though $M_b(t, \omega[t])$ and $M_f(t, \omega[t])$ are not semi-groups, following 'almost semi-group' properties hold (for part (1) see Griego/Hersh[10], for part (2) Keeper[20])

Proposition 2.14

For $s, t \geq 0$ we get

$$(1) \quad M_b(0, \omega[0]) = id_H \quad \text{and} \quad M_b(s+t, \omega[s+t]) = M_b(s, \omega[s]) M_b(t, \tilde{\omega}_s[t])$$

$$(2) \quad M_f(0, \omega[0]) = id_H \quad \text{and} \quad M_f(s+t, \omega[s+t]) = M_f(t, \tilde{\omega}_s[t]) M_f(s, \omega[s])$$

Proof:

- From the definitions immediately follow that $M_b(0, \omega[0]) = id_H$ and $M_f(0, \omega[0]) = id_H$.

- Because of $\tau_{N(s)} \leq s < \tau_{N(s)+1}$ and the semi-group properties of T_j it is

$$T_{\sigma_{N(s)}}(\tau_{N(s)+1} \Leftrightarrow \tau_{N(s)}) = T_{\sigma_{N(s)}}(s \Leftrightarrow \tau_{N(s)})T_{\sigma_{N(s)}}(\tau_{N(s)+1} \Leftrightarrow s),$$

and then

$$\begin{aligned} M_b(s+t, \boldsymbol{\omega}[s+t]) &= \\ &= T_{\sigma_0}(\tau_1) \cdots T_{\sigma_{N(s)}}(\tau_{N(s)+1} \Leftrightarrow \tau_{N(s)}) \cdots T_{\sigma_{N(s+t)}}(t \Leftrightarrow \tau_{N(s+t)}) \\ &= T_{\sigma_0}(\tau_1) \cdots T_{\sigma_{N(s)}}(s \Leftrightarrow \tau_{N(s)}) \cdot \\ &\quad \cdot T_{\sigma_{N(s)}}(\tau_{N(s)+1} \Leftrightarrow s) \cdots T_{\sigma_{N(s+t)}}(t \Leftrightarrow \tau_{N(s+t)}) \\ &= M_b(s, \boldsymbol{\omega}[s])T_{\sigma_{N(s)}}(\tau_{N(s)+1} \Leftrightarrow s) \cdots T_{\sigma_{N(s+t)}}(t \Leftrightarrow \tau_{N(s+t)}), \end{aligned}$$

so that with the definition of the shifted trajectory, (2.9), we get

$$\begin{aligned} M_b(s+t, \boldsymbol{\omega}[s+t]) &= M_b(s, \boldsymbol{\omega}[s])T_{\tilde{\sigma}_0}(\tilde{\tau}_0)T_{\tilde{\sigma}_1}(\tilde{\tau}_2 \Leftrightarrow \tilde{\tau}_1) \cdots T_{\tilde{\sigma}_{N_s(t)}}(t \Leftrightarrow \tilde{\tau}_{N_s(t)}) \\ &= M_b(s, \boldsymbol{\omega}[s])M_b(t, \tilde{\boldsymbol{\omega}}_s[t]). \end{aligned}$$

Analogous calculations yield the second equation in (2). □

2.2.2 The expectation semi-group

Definition 2.15

Let $(\mathcal{H}, \langle \cdot, \cdot \rangle) = \prod_{k=1}^n (H, [\cdot, \cdot]_k)$ be the cartesian product of the Hilbert spaces $(H, [\cdot, \cdot]_k)$ and let $\tilde{f} = (f_1, \dots, f_n)^t \in \mathcal{H}$ with $f_k \in H$. On \mathcal{H} are defined, component-wise at some time $t \geq 0$,

- (1) the **backward expectation operators** $\mathcal{E}_b(t)$ (Griego/Hersh[9] and [10])

$$(\mathcal{E}_b(t)\tilde{f})_k = E \left[M_b(t, \boldsymbol{\omega}[t])f_{\omega(t)} ; \omega(0) = k \right], \quad k = 1, \dots, n,$$

- (2) and the **forward expectation operators** $\mathcal{E}_f(t)$ (Kepler[18] and [20])

$$(\mathcal{E}_f(t)\tilde{f})_k = E \left[M_f(t, \boldsymbol{\omega}[t])f_{\omega(0)} ; \omega(t) = k \right], \quad k = 1, \dots, n.$$

Remark 2.16

- (1) In the paper Kepler[18] are given the definitions of the expectation operators in terms of integrals and in a recursive manner, it is also shown that these definitions coincide with the ones given above.
- (2) With the notation $E[M_b(t, \boldsymbol{\omega}[t])f_{\omega(t)} ; \omega(0) = k]$ is meant that the sum in the expectation value of $M_b(t, \boldsymbol{\omega}[t])f_{\omega(t)}$ shall be made over all trajectories $\boldsymbol{\omega}$ with initial state $\omega(0) = k$. The analogous is meant in the other case.

- (3) The definition of the forward expectation operators is more natural for our purposes in the following sense:

Imagine one knows the initial vector $y(0)$ and the initial distribution $p(0)$ of the Markov chain. Then with $f_j = p_j(0)y(0), j = 1, \dots, n$, the vector

$$\hat{\Psi}[y, p](t) := \sum_{k=1}^n (\mathcal{E}_f(t)\tilde{f})_k \in H$$

is the expected value of the solution to (2.13) at time t .

The following two theorems are due to R.Griego and R.Hersh [10].

Theorem 2.17

The family $\{\mathcal{E}_b(t) : t \geq 0\}$ of operators forms a C_0 -semi-group of bounded operators, i.e. it satisfies the conditions

- (1) $\exists L \geq 1, \alpha \geq 0 : \|\mathcal{E}_b(t)\|_{\mathcal{H}} \leq L e^{\alpha t}$
- (2) $\mathcal{E}_b(0) = id_{\mathcal{H}}$
- (3) $\mathcal{E}_b(s+t) = \mathcal{E}_b(s)\mathcal{E}_b(t) = \mathcal{E}_b(t)\mathcal{E}_b(s) \quad \text{for } s, t \geq 0$
- (4) $\forall \tilde{f} \in \mathcal{H} : \lim_{t \rightarrow 0} \|\mathcal{E}_b(t)\tilde{f} - \tilde{f}\|_{\mathcal{H}} = 0$

Proof:

- (1) Set $M_k := M \max_j |k_j|$, then we get with (2.8) and Lemma 2.12(1)

$$\begin{aligned} \|(\mathcal{E}_b(t)\tilde{f})_k\|_k &\leq E \left[\|M_b(t, \omega[t])f_{\omega(t)}\|_k; \omega(0) = k \right] \\ &\leq E \left[\|M_b(t, \omega[t])\|_k \|f_{\omega(t)}\|_k; \omega(0) = k \right] \\ &= \sum_{m=0}^{\infty} E \left[\|M_b(t, \omega[t])\|_k \|f_{\omega(t)}\|_k; \omega(0) = k \mid N(t) = m \right] \Pr(N(t) = m) \\ &\leq \sum_{m=0}^{\infty} M_k^{m+1} e^{\beta t} \|f\|_{\mathcal{H}} \frac{(c_1 t)^m}{m!} e^{-c_2 t} \\ &\leq M_k e^{(\beta + M_k c_1 - c_2)t} \|f\|_{\mathcal{H}}, \end{aligned}$$

then with $\tilde{M} := \max_k M_k$

$$\begin{aligned} \|\mathcal{E}_b(t)\tilde{f}\|_{\mathcal{H}}^2 &= \sum_{k=1}^n \|(\mathcal{E}_b(t)\tilde{f})_k\|_k^2 \\ &\leq n \tilde{M}^2 e^{2(\beta + \tilde{M} c_1 - c_2)t} \|\tilde{f}\|_{\mathcal{H}}^2 =: L^2 e^{2\alpha t} \|\tilde{f}\|_{\mathcal{H}}^2, \end{aligned} \tag{2.14}$$

and hence

$$\|\mathcal{E}_b(t)\|_{\mathcal{H}} \leq L e^{\alpha t}.$$

(2) Let $k \in \{1, \dots, n\}$, then

$$\begin{aligned} (\mathcal{E}_b(0)\tilde{f})_k &= E[M_b(0, \boldsymbol{\omega}[0])f_{\omega(0)}; \omega(0) = k] \\ &= E[T_{\omega(0)}f_{\omega(0)}; \omega(0) = k] = f_k \\ \Rightarrow \mathcal{E}_b(0) &= \text{id}_{\mathcal{H}}. \end{aligned}$$

(3) We have with the fact that $E[X] = E[E[X|Y]]$, the Markov property (2.3) and Proposition 2.14

$$\begin{aligned} (\mathcal{E}_b(s+t)\tilde{f})_k &= E[M_b(s+t, \boldsymbol{\omega}[s+t])f_{\omega(s+t)}; \omega(0) = k] \\ &= E\left[E[M_b(s+t, \boldsymbol{\omega}[s+t])f_{\omega(s+t)}; \omega(0) = k | \omega(u), 0 \leq u \leq s]; \omega(0) = k\right] \\ &= E\left[M_b(s, \boldsymbol{\omega}[s])E[M_b(t, \tilde{\boldsymbol{\omega}}_s[t])f_{\tilde{\omega}_s[t]}; \tilde{\omega}_s(0) = \omega(s)]; \omega(0) = k\right] \\ &= E\left[M_b(s, \boldsymbol{\omega}[s])(\mathcal{E}_b\tilde{f})_{\omega(s)}; \omega(0) = k\right] \\ &= (\mathcal{E}_b(s)\mathcal{E}_b(t)\tilde{f})_k \end{aligned}$$

(4) Again we use (2.8) and Lemma 2.12(1)

$$\begin{aligned} \|(\mathcal{E}_b(t)\tilde{f})_k \Leftrightarrow f_k\|_k &= \|E[M_b(t, \boldsymbol{\omega}[t])f_{\omega(t)}; \omega(0) = k] \Leftrightarrow f_k\|_k \\ &\leq E[\|M_b(t, \boldsymbol{\omega}[t])f_{\omega(t)} \Leftrightarrow f_k\|_k; \omega(0) = k] \\ &= \sum_{m=0}^{\infty} E\left[\|M_b(t, \boldsymbol{\omega}[t])f_{\omega(t)} \Leftrightarrow f_k\|_k; \omega(0) = k | N(t) = m\right] \Pr(N(t) = m) \\ &\leq \|T_k(t)f_k \Leftrightarrow f_k\|_k + \\ &\quad + \sum_{m=1}^{\infty} E\left[\|M_b(t, \boldsymbol{\omega}[t])f_{\omega(t)}\|_k \|f_k\|_k; \omega(0) = k | N(t) = m\right] \frac{(c_1 t)^m}{m!} e^{-c_2 t} \\ &\leq \|T_k(t)f_k \Leftrightarrow f_k\|_k + \|\tilde{f}\|_{\mathcal{H}} \sum_{m=1}^{\infty} (M_k^{m+1} e^{\beta t} + 1) \frac{(c_1 t)^m}{m!} e^{-c_2 t} \\ &= \|T_k(t)f_k \Leftrightarrow f_k\|_k + \|\tilde{f}\|_{\mathcal{H}} e^{-c_2 t} [M_k e^{\beta t} (e^{M_k c_1 t} \Leftrightarrow 1) + (e^{c_1 t} \Leftrightarrow 1)]. \end{aligned}$$

Since T_k are C_0 -semi-groups we have from above

$$\begin{aligned} \lim_{t \rightarrow 0} \|(\mathcal{E}_b(t)\tilde{f})_k \Leftrightarrow f_k\|_k &\leq \lim_{t \rightarrow 0} \left(\|T_k(t)f_k \Leftrightarrow f_k\|_k + \right. \\ &\quad \left. + \|\tilde{f}\|_{\mathcal{H}} e^{-c_2 t} [M_k e^{\beta t} (e^{M_k c_1 t} \Leftrightarrow 1) + (e^{c_1 t} \Leftrightarrow 1)] \right) \\ &= 0 \end{aligned}$$

□

Remark 2.18

Note that $\beta + \tilde{M}c_1 \Leftrightarrow c_2 \geq \beta$ in equation (2.14) since $\tilde{M} \geq 1$ and $c_1 \geq c_2$.

Theorem 2.19

The Cauchy problem

$$\begin{cases} \frac{1}{i} \frac{d}{dt} (\tilde{f})_j = A_j f_j \Leftrightarrow i \sum_{k=1}^n q_{jk} f_k, & j = 1, \dots, n \\ \tilde{f}(0) = \tilde{f}_0 \end{cases}$$

for a vector function $\tilde{f}(t) = (f_1(t), \dots, f_n(t))^t$, $t \geq 0$, on \mathcal{H} is solved by

$$\tilde{f}(t) = \mathcal{E}_b(t) \tilde{f}_0$$

That is, the infinitesimal generator \mathcal{A}_b of the backward expectation semi-group \mathcal{E}_b is

$$\begin{aligned} \mathcal{A}_b &:= \begin{pmatrix} A_1 & & 0 \\ & \ddots & \\ 0 & & A_n \end{pmatrix} \Leftrightarrow i \begin{pmatrix} q_{11} \text{id}_H & \cdots & q_{1n} \text{id}_H \\ \vdots & \ddots & \vdots \\ q_{n1} \text{id}_H & \cdots & q_{nn} \text{id}_H \end{pmatrix} = \\ &= \text{diag}(A_1, \dots, A_n) \Leftrightarrow i Q \times \text{id}_H \end{aligned}$$

Proof:

We show that \mathcal{A}_b generates \mathcal{E}_b .

Let τ be the time of the first jump of the Markov chain, and the index j at $\text{Pr}_j(X)$ denote that the chain is in state j . We have for $j \in \{1, \dots, n\}$

$$\begin{aligned} (\mathcal{E}_b(t) \tilde{f})_j &= E[M_b(t, \boldsymbol{\omega}[t]) f_{\omega(t)}; \omega(0) = j, \tau > t] + E[M_b(t, \boldsymbol{\omega}[t]) f_{\omega(t)}; \omega(0) = j, \tau \leq t] \\ &= T_j(t) f_j \text{Pr}_j(\tau > t) + \int_0^t E[M_b(t, \boldsymbol{\omega}[t]) f_{\omega(t)}; \omega(0) = j | \tau = r] \text{Pr}_j(\tau \in dr). \end{aligned}$$

Now with (2.1), (2.6), (2.7), and $\boldsymbol{\mu} := \tilde{\boldsymbol{\omega}}_r$

$$\begin{aligned} &\int_0^t E[M_b(t, \boldsymbol{\omega}[t]) f_{\omega(t)}; \omega(0) = j | \tau = r] \text{Pr}_j(\tau \in dr) \\ &= \int_0^t E[T_j(r) M_b(t \Leftrightarrow r, \tilde{\boldsymbol{\omega}}_r[t \Leftrightarrow r]) f_{\tilde{\omega}_r(t-r)}; \omega(0) = j | \tau = r] \text{Pr}_j(\tau \in dr) \\ &= \int_0^t T_j(r) \sum_{k=1, k \neq j}^n E[M_b(t \Leftrightarrow r, \boldsymbol{\mu}[t \Leftrightarrow r]) f_{\mu(t-r)}; \mu(0) = k] \text{Pr}_j(\omega(r) = k) \text{Pr}_j(\tau \in dr) \\ &= \int_0^t T_j(r) \sum_{k=1, k \neq j}^n (\mathcal{E}_b(t \Leftrightarrow r) \tilde{f})_k (q_{jk} e^{-q_j r}) dr. \end{aligned}$$

Then we have for the operator \mathcal{A}_b with Theorem A.23(1)

$$\begin{aligned} (\mathcal{A}_b \tilde{f})_j &= \lim_{t \rightarrow 0} \frac{(\mathcal{E}_b(t) \tilde{f})_j \Leftrightarrow f_j}{it} \\ &= \lim_{t \rightarrow 0} \frac{1}{it} \left[T_j(t) f_j \text{Pr}_j(\tau > t) \Leftrightarrow f_j + \int_0^t T_j(r) \sum_{k=1, k \neq j}^n (\mathcal{E}_b(t \Leftrightarrow r) \tilde{f})_k q_{jk} e^{-q_j r} dr \right] \\ &= \lim_{t \rightarrow 0} \left[\text{Pr}_j(\tau > t) \frac{T_j(t) f_j \Leftrightarrow f_j}{it} + \frac{\text{Pr}_j(\tau > t) \Leftrightarrow 1}{it} f_j + \right. \\ &\quad \left. + (\Leftrightarrow i) \sum_{k=1, k \neq j}^n q_{jk} \frac{1}{t} \int_0^t T_j(r) (\mathcal{E}_b(t \Leftrightarrow r) \tilde{f})_k e^{-q_j r} dr \right] \end{aligned}$$

and thus

$$\begin{aligned}
(\mathcal{A}_b \tilde{f})_j &= A_j f_j \Leftrightarrow i q_{jj} f_j \Leftrightarrow i \sum_{k=1, k \neq j}^n q_{jk} T_j(0) (\mathcal{E}_b(0) \tilde{f})_k \\
&= A_j f_j \Leftrightarrow i \sum_{k=1}^n q_{jk} f_k.
\end{aligned}$$

□

Remark 2.20

In the original papers Griego/Hersh[9] and [10] the result is stated that the generator \mathcal{A}_b of $\mathcal{E}_b(t)$ is $\mathcal{A}_b = \text{diag}(A_k) + Q \times \text{id}_H$, with Q instead of $(\Leftrightarrow i)Q$. The reason for the difference is that by their definition the semi-group generated by an operator A is e^{At} .

There is a simple connection between the backward and forward expectation semi-groups, also treated in Keepler[18] and in particular in Keepler[19].

Proposition 2.21

Let $\mathcal{E}_b(t)$ be the backward expectation semi-group of the n -state Markov chain ν with generator Q , and let $\mathcal{F}_f(t)$ be the forward expectation semi-group of the n -state Markov chain η with the transposed generator Q^t . Then $\mathcal{E}_b(t) = \mathcal{F}_f(t)$. In particular we have

$$Q = Q^t \quad \Rightarrow \quad \mathcal{E}_b(t) = \mathcal{E}_f(t).$$

Proof:

Let $M_b(t, \boldsymbol{\omega}[t])$ be the backward random evolution for ν and $M_f(t, \boldsymbol{\mu}[t])$ the forward random evolution of η . Further let $E_\nu[X]$ be the expected value corresponding to the chain ν , analogous for $E_\eta[X]$. Then for $k = 1, \dots, n$ and $\tilde{f} \in \mathcal{H}$

$$\begin{aligned}
(\mathcal{E}_b(t) \tilde{f})_k &= E_\nu [M_b(t, \boldsymbol{\omega}[t]) f_{\omega(t)}; \omega(0) = k] \\
&= \sum_{j=1}^n E_\nu [M_b(t, \boldsymbol{\omega}[t]) f_j; \omega(0) = k \wedge \omega(t) = j] \\
&= E_\nu [M_f(t, \hat{\boldsymbol{\omega}}[t]) f_{\hat{\omega}(0)}; \hat{\omega}(t) = k].
\end{aligned} \tag{2.15}$$

Now, the probability of the reversed trajectory $\hat{\boldsymbol{\omega}}$ for ν is the same as the probability of the ‘forward’ trajectory $\boldsymbol{\mu} = \hat{\boldsymbol{\omega}}$ for η , since η is generated by Q^t and Lemma 2.10(2). Thus we get in (2.15)

$$(\mathcal{E}_b(t) \tilde{f})_k = (\mathcal{F}_f(t) \tilde{f})_k.$$

□

Corollary 2.22

The family $\{\mathcal{E}_f(t) : t \geq 0\}$ forms a C_0 -semi-groups of operators with infinitesimal generator

$$A_f := \text{diag}(A_1, \dots, A_n) \Leftrightarrow i Q^t \times \text{id}_H$$

Remark 2.23

Regarding the operator $Q \times \text{id}_H$ on \mathcal{H} we make following observations:

- (1) The operators id_H represent the identity operator in H . Should H be of finite dimension, and the operators A_j be represented as some matrices with respect to different bases, then the operators id_H are not represented as identity matrices but rather as some change of basis matrices, according to their position in the matrix operator $Q \times \text{id}_H$.
- (2) In the case that the norms $\|\cdot\|_1, \dots, \|\cdot\|_n$ are different the operator id_H in the j th row and k th column of $Q \times \text{id}_H$ is the embedding

$$I_{jk} : (H, \langle \cdot, \cdot \rangle_k) \rightarrow (H, \langle \cdot, \cdot \rangle_j), \quad u \mapsto u,$$

and the operator norm of I_{jk} is not necessarily equal to 1.

2.3 Stability in Average

Our concept of stability is motivated from the physical context where the (suitably chosen) norm of a vector function $u(t)$ may represent the physical energy of a certain object, e.g. a particle or a string. We call an evolution given by a semi-group $\{T(t)\}_{t \geq 0}$ of bounded operators stable if for each $u \in H$ the norms $\|T(t)u\|_H$ are uniformly bounded (or equi-bounded) over all times $t \geq 0$. In general the matrix Q will not be symmetric, but then the forward and backward expectation semi-groups are not equal. Considering their definitions and Remark 2.16(3) it seems more natural to connect stability in average with the forward evolution, thus we suggest the

Definition 2.24

The dynamical system given as the solution of the Cauchy problem (2.13) is said to be **stable in average** if the operators $\mathcal{E}_f(t)$ of the forward expectation semi-group are uniformly bounded, i.e. if there exists a constant $K > 0$, such that $\|\mathcal{E}_f(t)\|_{\mathcal{H}} \leq K$ for all $t \geq 0$.

Remark 2.25

- (1) It would be interesting to find an example of a system (with non-symmetric Q) where the forward expectation semi-group is uniformly bounded, and the backward is not, or vice versa.
- (2) If the operator norms of the family $\{\mathcal{E}_f(t)\}_{t \geq 0}$ are uniformly bounded, then the norms of the expected values $\Psi[f, p](t)$ are uniformly bounded for every initial vector $f \in H$ and initial distribution p , this corresponds to the described concept of stability.
- (3) Certainly, if the generator of the expectation semi-group is dissipative then the system is stable in average (cf. Appendix A.2).

Before we state some sufficient and necessary condition for strong dissipativity of the operator \mathcal{A}_f , we list some properties of the matrix operator $Q \times \text{id}_H$.

Lemma 2.26

If all inner products are mutually equivalent then $Q \times \text{id}_H$ is a bounded operator on \mathcal{H} .

Conversely, if Q generates an irreducible Markov process, and if $Q \times \text{id}_H$ is bounded, then all inner products are mutually equivalent.

Proof:

Let $\gamma_1, \dots, \gamma_n$ and $\alpha_1, \dots, \alpha_n$ be positive constants, such that

$$\gamma_j^{-1} \|f\|_j^2 \leq \|f\|_1^2 \leq \alpha_j^{-1} \|f\|_j^2$$

for all $f \in H$ and $j = 1, \dots, n$, and let $\tilde{f} = (f_1, \dots, f_n)^t \in \mathcal{H}$. Then we get

$$\begin{aligned} \|Q \times \text{id}_H \tilde{f}\|^2 &= \sum_{j=1}^n \left\| \sum_{k=1}^n q_{jk} f_k \right\|_j^2 \leq \sum_{j=1}^n \left(\sum_{k=1}^n |q_{jk}| \|f_k\|_j \right)^2 \\ &\leq \sum_{j=1}^n n \left(\sum_{k=1}^n |q_{jk}|^2 \|f_k\|_j^2 \right) \leq n \max_{j,k} (|q_{jk}|^2) \sum_{j=1}^n \sum_{k=1}^n \gamma_j \|f_k\|_1^2 \\ &\leq n^2 \max_{j,k} (|q_{jk}|^2) \max_j (\gamma_j) \sum_{k=1}^n \|f_k\|_1^2 \end{aligned}$$

and also

$$\|\tilde{f}\|^2 = \sum_{k=1}^n \|f_k\|_k^2 \geq \sum_{k=1}^n \alpha_k \|f_k\|_1^2 \geq \min_k (\alpha_k) \sum_{k=1}^n \|f_k\|_1^2$$

Now we get

$$\frac{\|Q \times \text{id}_H \tilde{f}\|^2}{\|\tilde{f}\|^2} \leq n^2 \max_{j,k} (|q_{jk}|^2) \frac{\max_j (\gamma_j)}{\min_k (\alpha_k)}$$

so that $Q \times \text{id}_H$ is bounded.

For the converse we assume that the norms $\|\cdot\|_j$ for $j \in J \subsetneq S = \{1, \dots, n\}$, and the norms $\|\cdot\|_k$ for $k \in K := S \setminus J$ are mutually equivalent respectively, whereas $\|\cdot\|_j$ and $\|\cdot\|_k$ are not equivalent for $j \in J$ and $k \in K$.

Since the Markov process is irreducible, there exist $j_0 \in J$ and $k_0 \in K$ such that $q_{j_0 k_0} \neq 0$. (cf. Grimmett/Stirzaker[11]), w.l.o.g. let $j_0 = 2, k_0 = 1$.

Now, $\|\cdot\|_1$ and $\|\cdot\|_2$ are not equivalent, so that

$$\forall m > 0 \quad \exists f_m \in H : \|f_m\|_2^2 > m \|f_m\|_1^2.$$

We set $\tilde{f}_m = (f_m, 0, \dots, 0)^t \in \mathcal{H}$, then we have with $\|\tilde{f}_m\|_{\mathcal{H}}^2 = \|f_m\|_1^2$

$$\begin{aligned} \|Q \times \text{id}_H \tilde{f}_m\|_{\mathcal{H}}^2 &= \|(q_{11} f_m, \dots, q_{n1} f_m)^t\|_{\mathcal{H}}^2 = \sum_{j=1}^n |q_{j1}|^2 \|f_m\|_j^2 \\ &\geq |q_{21}|^2 \|f_m\|_2^2 > m |q_{21}|^2 \|\tilde{f}_m\|_{\mathcal{H}}^2 \end{aligned}$$

Hence $Q \times \text{id}_H$ is unbounded, contrary to the assumption.

□

Lemma 2.27

Let Q be a complex $n \times n$ -matrix.

If all inner products $\langle \cdot, \cdot \rangle_1, \dots, \langle \cdot, \cdot \rangle_n$ are equal, then the adjoint of $Q \times \text{id}_H$ in $(\mathcal{H}, [\cdot, \cdot])$ is $\bar{Q}^t \times \text{id}_H$.

Conversely, if Q generates an irreducible n -state Markov process and $(Q \times \text{id}_H)^* = (\bar{Q}^t \times \text{id}_H)$, then all inner products are equal.

Proof:

Let $Q = (q_{jk})_{j,k=1}^n, \bar{Q}^t = (\tilde{q}_{jk})_{j,k=1}^n = (\overline{q_{kj}})_{j,k=1}^n, U = (u_1, \dots, u_n)^t$ and $V = (v_1, \dots, v_n)^t \in \mathcal{H}$.

Suppose $\langle \cdot, \cdot \rangle_1 = \dots = \langle \cdot, \cdot \rangle_n =: \langle \cdot, \cdot \rangle$, then we get

$$\begin{aligned} [Q \times \text{id}_H U, V] &= \sum_{j=1}^n \left\langle \sum_{k=1}^n q_{jk} u_k, v_j \right\rangle = \sum_{k=1}^n \left\langle u_k, \sum_{j=1}^n \overline{q_{jk}} v_j \right\rangle \\ &= [U, \bar{Q}^t \times \text{id}_H V] \end{aligned}$$

so that the adjoint of $Q \times \text{id}_H$ is $\bar{Q}^t \times \text{id}_H$.

For the converse, we have

$$[Q \times \text{id}_H U, V] = \sum_{j=1}^n \sum_{k=1}^n q_{jk} \langle u_k, v_j \rangle_j \tag{2.16}$$

$$[U, \bar{Q}^t \times \text{id}_H V] = \sum_{j=1}^n \sum_{k=1}^n \langle u_j, \tilde{q}_{jk} v_k \rangle_j \tag{2.17}$$

Equations (2.16) and (2.17) are equal if and only if

$$\sum_{j=1}^n \sum_{k=1}^n q_{jk} \langle u_k, v_j \rangle_j = \sum_{j=1}^n \sum_{k=1}^n q_{kj} \langle u_j, v_k \rangle_j \tag{2.18}$$

for all $U, V \in \mathcal{H}$. If we choose $u_k = f \delta_{lk}$ and $v_j = g \delta_{mj}$ with $m, l = 1, \dots, n$ and $f, g \in H$, then (2.18) yields

$$q_{ml} \langle f, g \rangle_m = q_{ml} \langle f, g \rangle_l \quad \forall f, g \in H.$$

If $q_{ml} \neq 0$, then we immediately get $\langle \cdot, \cdot \rangle_m = \langle \cdot, \cdot \rangle_l$. If $q_{ml} = 0$ we use a similar argument as in Lemma 2.26, since the Markov process is assumed irreducible, and thus we obtain that $\langle \cdot, \cdot \rangle_m = \langle \cdot, \cdot \rangle_l$ for all $m, l = 1, \dots, n$.

□

Corollary 2.28

If all inner products are equal, then $\text{Re}(Q \times \text{id}_H) = (\text{Re } Q) \times \text{id}_H$ and $\text{Im}(Q \times \text{id}_H) = (\text{Im } Q) \times \text{id}_H$.

Before we state the next Lemma we note that each element (y, g) of the space $\mathbb{C} \times H$ can be identified with the vector $(y_1g, \dots, y_n g)^t \in \mathcal{H}$.

Lemma 2.29

Assume all inner products are equal. If there exists a subspace $Y \subset \mathbb{C}^n$ and $m \in \mathbb{R}$ such that $\langle Qx, x \rangle_{\mathbb{C}} \leq m \|x\|_{\mathbb{C}}^2$ for all $x \perp_{\mathbb{C}} Y$, then we have

$$[Q \times \text{id}_H U, U] \leq m[U, U] \quad \forall U \perp_{\mathcal{H}} (Y \times H)$$

Proof:

From the assumption we have that for all $x \perp_{\mathbb{C}} Y$

$$\langle Qx, x \rangle_{\mathbb{C}} = \sum_{j,k=1}^n q_{jk} x_k \bar{x}_j \leq \lambda_2 \|x\|_{\mathbb{C}}^2 \quad (2.19)$$

A vector $U = (u_1, \dots, u_n)^t \in \mathcal{H}$ is orthogonal to $Y \times H$ if and only if

$$\forall y \in Y, g \in H \quad [U, yg] = \sum_{j=1}^n \langle u_j, y_j g \rangle_H = 0 \quad \Leftrightarrow \quad \sum_{j=1}^n u_j \bar{y}_j = 0 \quad \forall y \in Y \quad (2.20)$$

Let $\{\phi_l\}_{l \in \mathbb{N}}$ be an orthonormal basis for H , so that for each $j = 1, \dots, n$ exist coefficients $\{\sigma_j^l\}_{l \in \mathbb{N}}$ with $u_j = \sum_{l \in \mathbb{N}} \sigma_j^l \phi_l$. Now we get

$$\begin{aligned} [Q \times \text{id}_H U, U]_{\mathcal{H}} &= \sum_{j=1}^n \left\langle \sum_{k=1}^n q_{jk} u_k, u_j \right\rangle_H = \sum_{j=1}^n \sum_{k=1}^n q_{jk} \left\langle \sum_{l \in \mathbb{N}} \sigma_k^l \phi_l, \sum_{m \in \mathbb{N}} \sigma_j^m \phi_m \right\rangle_H \\ &= \sum_{l \in \mathbb{N}} \left(\sum_{j=1}^n \sum_{k=1}^n q_{jk} \sigma_k^l \bar{\sigma}_j^l \right) \end{aligned} \quad (2.21)$$

We define $x_l = (\sigma_1^l, \dots, \sigma_n^l)^t$ for $l \in \mathbb{N}$, so that from (2.21) follows

$$[Q \times \text{id}_H U, U]_{\mathcal{H}} = \sum_{l \in \mathbb{N}} \langle Qx_l, x_l \rangle_{\mathbb{C}} \quad (2.22)$$

If $U \perp_{\mathcal{H}} Y \times H$ then $x_l \perp_{\mathbb{C}} Y$ for all $l \in \mathbb{N}$, since from (2.20) it follows with $y \in Y$

$$\begin{aligned} 0 &= \sum_{j=1}^n \bar{y}_j u_j = \sum_{j=1}^n y_j \sum_{l \in \mathbb{N}} \sigma_j^l \phi_l = \sum_{l \in \mathbb{N}} \left(\sum_{j=1}^n \bar{y}_j \sigma_j^l \right) \phi_l \\ \Leftrightarrow \quad 0 &= \sum_{j=1}^n \bar{y}_j \sigma_j^l = \langle x_l, y \rangle_{\mathbb{C}} \quad \forall l \in \mathbb{N}, y \in Y \end{aligned}$$

Finally, from (2.19) and (2.22) follows

$$[Q \times \text{id}_H U, U]_{\mathcal{H}} \leq \lambda_2 \sum_{l \in \mathbb{N}} \|x_l\|_{\mathbb{C}}^2 = \lambda_2 \sum_{l \in \mathbb{N}} \sum_{j=1}^n |\sigma_j^l|^2 = \lambda_2 \sum_{j=1}^n \|u_j\|_H^2 = \lambda_2 \|U\|_{\mathcal{H}}^2$$

□

Corollary 2.30

If all inner products are equal, and if $0 = \lambda_1 > \lambda_2 \geq \dots \geq \lambda_n$ are the eigenvalues of the real and symmetric matrix Q , where $\lambda_1 = 0$ is simple with eigenvector π^t , then

$Q \times id_H$ is bounded from above on the orthogonal complement of $\{\pi^t\} \times H$ in \mathcal{H} , i.e. with $\mathcal{R} := \mathcal{H} \ominus (\{\pi^t\} \times H)$

$$Q \times id_H|_{\mathcal{R}} \leq \lambda_2 id_{\mathcal{R}}$$

Now we are set to investigate conditions for the strong dissipativity of the operator $\mathcal{A}_f(\varkappa) := (\text{diag}(A_1, \dots, A_n) \Leftrightarrow i\varkappa Q^t \times id_H)$ for the special case that the operators A_1, \dots, A_n are bounded and all inner products are equal. Inserting the variable $\varkappa > 0$ corresponds to 'accelerating' or 'decelerating' the Markov process generated by the matrix Q .

Theorem 2.31

Let the generator $A_{\nu(t)}$ of the dynamical system (2.13) jump between the operators A_1, \dots, A_n on the Hilbert space $(H, \langle \cdot, \cdot \rangle)$ via a Markov process which is generated by the symmetric matrix Q and which has the equilibrium distribution $\pi = (\pi_1, \dots, \pi_n)$. Assume that A_1, \dots, A_n are bounded operators such that

$$Im(A_j) \geq \mu_j id_H \quad (2.23)$$

for $j = 1, \dots, n$, and largest possible values of $\mu_1, \dots, \mu_n \in \mathbb{R}$.
Now, if for some $\alpha > 0$

$$Im\left(\sum_{j=1}^n \pi_j^2 A_j\right) \geq \alpha \left(\sum_{j=1}^n \pi_j^2\right) id_H \quad (2.24)$$

then there exist $0 < \lambda < \alpha$ and $\varkappa > 0$, such that

$$Im \mathcal{A}_f(\varkappa) \geq \lambda id_{\mathcal{H}}.$$

Conversely, if $Im \mathcal{A}_f(\varkappa) \geq \lambda I_{\mathcal{H}}$ for some $\lambda > 0$ and $\varkappa \in \mathbb{R}$, then $Im\left(\sum_{j=1}^n \pi_j^2 A_j\right) \geq \lambda \left(\sum_{j=1}^n \pi_j^2\right) id_H$.

Proof:

We define $\mu_0 := \min\{\mu_j : j = 1, \dots, n\}$, $\tilde{A} := \text{diag}(A_1, \dots, A_n)$ and $\sigma := \|Im \tilde{A}\| = \left(\sum_{j=1}^n \|Im(A_j)\|^2\right)^{1/2}$. It follows from (2.23) that $Im(\text{diag}(A_1, \dots, A_n)) \geq \mu_0 id_{\mathcal{H}}$, and note that σ is positive unless all operators A_j are selfadjoint, and this is the situation treated in Pavlov[28].

Q is a stochastic matrix, so that zero is a simple eigenvalue of Q^t with (right) eigenvector $\pi \in \mathbb{R}^n$, all other eigenvalues are negative. The nullspace of $M := \Leftrightarrow i Q^t \times id_H$ is

$$\mathcal{N} := \{(\pi_1 f, \dots, \pi_n f)^t : f \in H\} = \{\pi^t\} \times H.$$

Let $\mathcal{R} := \mathcal{H} \ominus \mathcal{N}$ and $m_0 := \min\{|\tau| : \tau \text{ is non-zero eigenvalue of } Q^t\}$, then with Corollary 2.30 we get

$$Im(M|_{\mathcal{R}}) = \Leftrightarrow Re(Q^t \times id_H|_{\mathcal{R}}) \geq m_0 id_{\mathcal{R}}$$

The adjoint of M is $\Leftrightarrow M$, since $Q = Q^t$, so it follows $M\mathcal{R} \perp \mathcal{N}$. Now, using above estimations and inequalities, we get with $V \in \mathcal{R}, W \in \mathcal{N}, U = V + W \in \mathcal{H}$

$$\begin{aligned} \operatorname{Im}[(\tilde{A} + \varkappa M)U, U] &= \\ &= \operatorname{Im}[(\tilde{A} + \varkappa M)V, V] + \operatorname{Im}[\tilde{A}W, V] + \operatorname{Im}[\tilde{A}V, W] + \operatorname{Im}[\tilde{A}W, W] \\ &= \operatorname{Im}[(\tilde{A} + \varkappa M)V, V] + \operatorname{Im}[(\tilde{A} \Leftrightarrow \tilde{A}^*)V, W] + \operatorname{Im}\left(\sum_{j=1}^n \langle A_j \pi_j f, \pi_j f \rangle\right) \end{aligned}$$

Thus we get with some $\varepsilon > 0$

$$\begin{aligned} \operatorname{Im}[(\tilde{A} + \varkappa M)U, U] &= \\ &\geq (\mu_0 + \varkappa m_0) \|V\|^2 \Leftrightarrow 2 \|(\tilde{A} \Leftrightarrow \tilde{A}^*)\| \|V\| \|W\| + \operatorname{Im}\left\langle \sum_{j=1}^n \pi_j^2 A_j f, f \right\rangle \\ &\geq (\mu_0 + \varkappa m_0) \|V\|^2 \Leftrightarrow \| \operatorname{Im}(\tilde{A}) \| \left(\varepsilon \|V\|^2 + \frac{1}{\varepsilon} \|W\|^2 \right) + \alpha \left(\sum_{j=1}^n \pi_j^2 \right) \|f\|_H^2 \\ &= (\mu_0 + \varkappa m_0 \Leftrightarrow \varepsilon \sigma) \|V\|^2 + \left(\alpha \Leftrightarrow \frac{\sigma}{\varepsilon} \right) \|W\|^2 = (*) \end{aligned}$$

We require that

$$(*) \geq \lambda \|U\|^2 = \lambda (\|V\|^2 + \|W\|^2),$$

which is satisfied if

$$\varkappa m_0 + \mu_0 \Leftrightarrow \varepsilon \sigma \geq \lambda \tag{2.25}$$

$$\alpha \Leftrightarrow \frac{\sigma}{\varepsilon} \geq \lambda. \tag{2.26}$$

Equation (2.26) is fulfilled for $\varepsilon = \frac{\sigma}{\alpha - \lambda}$ and any $0 < \lambda < \alpha$, and then (2.25) is satisfied for

$$\varkappa \geq \frac{1}{m_0} \left(\lambda \Leftrightarrow \mu_0 + \frac{\sigma^2}{\alpha \Leftrightarrow \lambda} \right) \tag{2.27}$$

For the converse, choose $U = \pi^t f \in \{\pi^t\} \times H$ and observe that

$$\operatorname{Im}\left\langle \sum_{j=1}^n \pi_j^2 A_j f \right\rangle_H = \operatorname{Im}[(\tilde{A} + \varkappa M)U, U] \geq \lambda \|U\|_H^2 = \lambda \left(\sum_{j=1}^n \pi_j^2 \right) \|f\|_H^2$$

This proves the theorem. □

The system is stable in average also in the case when \mathcal{A}_f is only similar to a strongly dissipative operator (see Proposition A.29, which is considered in the next theorem).

Theorem 2.32

In the situation of the theorem above let P_0 and P_1 be the orthogonal projections of \mathcal{H} onto \mathcal{N} and \mathcal{R} respectively.

Then the operator $\mathcal{A}_f(\varkappa)$ is similar to a strongly dissipative operator for some $\varkappa > 0$ if the component $P_0 A P_0$ is similar to a dissipative operator B_0 on $\mathcal{N} = \{\pi^t\} \times H$, ie. if there exists a bounded and invertible operator $X : \mathcal{N} \rightarrow \mathcal{N}$ and $\beta > 0$ such that $B_0 = X P_0 A P_0 X^{-1}$ and $\operatorname{Im}(B_0) \geq \beta \operatorname{id}_{\mathcal{N}}$.

Proof:

First we note that the operator $X : \mathcal{N} \rightarrow \mathcal{N}$ can be identified with an operator on H , since a vector $(\pi_1 f, \dots, \pi_n f)^t \in \mathcal{N}$ is mapped to a vector $(\pi_1 g, \dots, \pi_n g)^t \in \mathcal{N}$, thus the essential part of X is the mapping from $f \in H$ to $g \in H$. This operator we also denote with X .

We transform the operator $\mathcal{A}_f(\varkappa)$ on $\mathcal{R} \times \mathcal{N}$ to

$$\mathcal{B}_f(\varkappa) := \begin{pmatrix} \text{id}_{\mathcal{R}} & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} P_1(\tilde{A} + \varkappa M)P_1 & P_1\tilde{A}P_0 \\ P_0\tilde{A}P_1 & P_0\tilde{A}P_0 \end{pmatrix} \begin{pmatrix} \text{id}_{\mathcal{R}} & 0 \\ 0 & X^{-1} \end{pmatrix},$$

and by similar calculations as in the proof of Theorem 2.31 we can find

$$\varkappa \geq \frac{1}{m_0} \left(\lambda \Leftrightarrow \mu_0 + \frac{\sigma^2}{\beta \Leftrightarrow \lambda} \right)$$

with $0 < \lambda < \beta$ and $\sigma = \frac{1}{2}\|\tilde{A}\|(\|X\| + \|X^{-1}\|)$ so that

$$\text{Im}(\mathcal{B}_f(\varkappa)) \geq \lambda \text{id}_{\mathcal{H}}.$$

□

Remark 2.33

We can extend the conclusions of Theorem 2.31 and 2.32 at least to the case where Q is a real and normal matrix with one zero eigenvalue and which other eigenvalues have strictly negative real part, since then $\text{Re}Q$ fulfils the conditions of Lemma 2.29 and it is also $M\mathcal{R} \perp \mathcal{N}$.

Theorems 2.31 and 2.32 deal with the situation of equal norms. However, in many interesting situations, like the examples in Chapters 3, 4 and 5, the canonical choice of the norms $\|\cdot\|_j = \sqrt{\langle \cdot, \cdot \rangle_j}$ on each copy of H results in merely equivalent norms, and the Markov process generally is not symmetric. Additionally, it is sufficient for stability in average that the generator \mathcal{A}_f of the expectation semi-group be similar to a dissipative operator. Thus with view to Theorem 2.32 we pose the following conjecture.

Conjecture 2.34

Let A_1, \dots, A_n be bounded operators, satisfying (2.23), on some Hilbert spaces $(H, \langle \cdot, \cdot \rangle_1), \dots, (H, \langle \cdot, \cdot \rangle_n)$ with mutually equivalent inner products, and let Q be the generator of the Markov process with equilibrium distribution $\pi = (\pi_1, \dots, \pi_n)$. Then the following two statements are equivalent:

- (i) There exists a positive \varkappa such that the dynamical system (2.13) affected by the Markov process generated by $\varkappa Q$ is stable in average.
- (ii) The operator $\hat{A} := \pi_1^2 A_1 + \dots + \pi_n^2 A_n$ generates a semi-group $e^{i\hat{A}t}$ of uniformly bounded operators.

2.4 The generator \mathcal{A} of the expectation semi-group for a 2-state Markov chain

Here we consider the special case of a symmetric 2-state Markov chain ν with generator

$$Q := \varkappa \begin{pmatrix} \Leftrightarrow 1 & 1 \\ 1 & \Leftrightarrow 1 \end{pmatrix}$$

where the sojourn times in the states are exponentially distributed with intensity $\varkappa > 0$. Let A_1 and A_2 be two infinitesimal generators of C_0 -semi-groups T_1 and T_2 on a Hilbert space H with domains $\mathcal{D}(A_1)$ and $\mathcal{D}(A_2)$. Since Q is symmetric, the forward and backward expectation semi-groups are equal, i.e. $\mathcal{E}_b(t) = \mathcal{E}_f(t) =: \mathcal{E}(t)$. The generator of $\mathcal{E}(t)$ is

$$\mathcal{A}(\varkappa) = \begin{pmatrix} A_1 + i\varkappa \text{id}_H & \Leftrightarrow i\varkappa \text{id}_H \\ \Leftrightarrow i\varkappa \text{id}_H & A_2 + i\varkappa \text{id}_H \end{pmatrix}$$

We want to find conditions so that \mathcal{A} is dissipative or similar to a dissipative operator, so the spectrum of \mathcal{A} is of special interest, especially its dependence on \varkappa .

Remark 2.35

- (1) From Lemma 2.27 we know, that $\Leftrightarrow iQ \times \text{id}_H$ is not symmetric, so $\mathcal{A}(\varkappa)$ is not symmetric either.
- (2) If all norms are equivalent, it follows from Lemma 2.26 that the operator $\mathcal{A}(\varkappa)$ can be regarded as a bounded perturbation of the diagonal operator $\mathcal{A}(0) = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ for all $\varkappa > 0$.
- (3) From the proof of Lemma 2.26 we get an estimate for the operator norm of $\Leftrightarrow iQ^t \times \text{id}_H$

$$\varkappa \leq \| \Leftrightarrow iQ^t \times \text{id}_H \|_{\mathcal{H}} \leq 2\varkappa \sqrt{\frac{\gamma_1}{, 1}}$$

2.4.1 The Frobenius-Schur factorisation of \mathcal{A}

In order to study matrix operators we can express them as a Frobenius-Schur factorisation, also called LDR factorisation in books on Numerical Analysis, eg. Stoer/Bulirsch[32]. As one can check directly, we have for $\mathcal{A}(\varkappa) \Leftrightarrow \alpha \text{id}_{\mathcal{H}}$ with $\alpha \in \mathbb{C}$

$$\begin{aligned} \mathcal{A}(\varkappa) \Leftrightarrow \alpha \text{id}_{\mathcal{H}} &= \begin{pmatrix} \text{id}_{\mathcal{H}} & 0 \\ \Leftrightarrow i\varkappa R_1^{\alpha - i\varkappa} & \text{id}_H \end{pmatrix} \cdot \begin{pmatrix} A_1 \Leftrightarrow (\alpha \Leftrightarrow i\varkappa) \text{id}_H & 0 \\ 0 & A_2 \Leftrightarrow (\alpha \Leftrightarrow i\varkappa) \text{id}_H + \varkappa^2 R_1^{\alpha - i\varkappa} \end{pmatrix} \begin{pmatrix} \text{id}_H & \Leftrightarrow i\varkappa R_1^{\alpha - i\varkappa} \\ 0 & \text{id}_H \end{pmatrix} \end{aligned} \quad (2.28)$$

or

$$\begin{aligned} \mathcal{A}(\varkappa) \Leftrightarrow \alpha \text{id}_{\mathcal{H}} &= \begin{pmatrix} \text{id}_{\mathcal{H}} & \Leftrightarrow i\varkappa R_2^{\alpha - i\varkappa} \\ 0 & \text{id}_H \end{pmatrix} \cdot \begin{pmatrix} A_1 \Leftrightarrow (\alpha \Leftrightarrow i\varkappa) \text{id}_H + \varkappa^2 R_2^{\alpha - i\varkappa} & 0 \\ 0 & A_2 \Leftrightarrow (\alpha \Leftrightarrow i\varkappa) \text{id}_H \end{pmatrix} \begin{pmatrix} \text{id}_H & 0 \\ \Leftrightarrow i\varkappa R_2^{\alpha - i\varkappa} & \text{id}_H \end{pmatrix} \end{aligned} \quad (2.29)$$

where $R_1^{\alpha-i\kappa}$ and $R_2^{\alpha-i\kappa}$ are the resolvents to $\alpha \Leftrightarrow i\kappa$ of A_1 and A_2 respectively.

2.4.2 \mathcal{A} for small intensity

For small values of the intensity κ we can regard the generator $\mathcal{A}(\kappa)$ as a weak perturbation of the diagonal operator $\mathcal{A}(0)$. We assume here that A_1 and A_2 are self-adjoint or anti-self-adjoint.

Let α_0 be an eigenvalue of $\mathcal{A}(0)$ with eigenvector u_0 . For small κ we can expand the eigenvalue $\alpha(\kappa)$ to the eigenvector $u(\kappa)$ of $\mathcal{A}(\kappa)$ as power series

$$\begin{aligned}\alpha(\kappa) &= \alpha_0 + \kappa\alpha_1 + \dots \\ u(\kappa) &= u_0 + \kappa u_1 + \dots\end{aligned}$$

Then we get with $B := \begin{pmatrix} \text{id}_H & -\text{id}_H \\ -\text{id}_H & \text{id}_H \end{pmatrix}$

$$\begin{aligned}\mathcal{A}(\kappa)u(\kappa) &= (\mathcal{A}(0) + i\kappa B)(u_0 + \kappa u_1 + \dots) \\ &= \mathcal{A}(0)u_0 + \kappa(iBu_0 + \mathcal{A}(0)u_1) + \dots, \\ \text{and } \alpha(\kappa)u(\kappa) &= (\alpha_0 + \kappa\alpha_1 + \dots)(u_0 + \kappa u_1 + \dots) \\ &= \alpha_0 u_0 + \kappa(\alpha_1 u_0 + \alpha_0 u_1) + \dots\end{aligned}$$

Comparing the coefficients in above equations and dropping the higher order terms we get

$$\mathcal{A}(0)u_0 = \alpha_0 u_0 \quad \wedge \quad \mathcal{A}(0)u_1 + i\kappa B \text{id}_H u_0 = \alpha_0 u_1 + \alpha_1 u_0.$$

When we waive any condition on the norm of u_1 we can choose $u_1 \perp u_0$ and get

$$\langle (\mathcal{A}(0) \Leftrightarrow \alpha_0)u_1, u_0 \rangle = \alpha_1 \langle u_0, u_0 \rangle \Leftrightarrow i\kappa \langle Bu_0, u_0 \rangle.$$

If A_1 and A_2 are self-adjoint or anti-self-adjoint, we get $\langle (\mathcal{A}(0) \Leftrightarrow \alpha_0)u_1, u_0 \rangle = 0$ and then

$$\alpha_1 = i\kappa \frac{\langle Bu_0, u_0 \rangle}{\langle u_0, u_0 \rangle}. \tag{2.30}$$

Since the eigenvectors u_0 of $\mathcal{A}(0) = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ are of the form $(v_1, 0)^t$ (or $(0, v_2)^t$), where v_1 (or v_2) is eigenvector of A_1 (or A_2), we have for instance

$$\langle Bu_0, u_0 \rangle = \left\langle \begin{pmatrix} \text{id}_H & \Leftrightarrow \text{id}_H \\ \Leftrightarrow \text{id}_H & \text{id}_H \end{pmatrix} \begin{pmatrix} v_1 \\ 0 \end{pmatrix}, \begin{pmatrix} v_1 \\ 0 \end{pmatrix} \right\rangle = \langle v_1, v_1 \rangle$$

and $\langle u_0, u_0 \rangle = \langle v_1, v_1 \rangle$.

From (2.30) follows that $\alpha_1 = i\kappa$, similar for the other case $u_0 = (0, v_2)^t$.

This actually means, that as a linear approximation of the eigenvalues of $\mathcal{A}(\kappa)$ at $\kappa = 0$ the imaginary parts increase with increasing value of κ , that is, graphically, the eigenvalues in the complex plane “go up”.

2.4.3 The adjoint operator of \mathcal{A}

IN A HILBERT SPACE OF FINITE DIMENSION

As a first step assume that A_1 and A_2 are linear operators on a finite dimensional Hilbert space $H = \mathbb{C}^n$, i.e. $n \times n$ -matrices. Then the hermitian of $\mathcal{A}(\varkappa)$ is

$$\bar{\mathcal{A}}^t(\varkappa) = \begin{pmatrix} \bar{A}_1^t \Leftrightarrow i\varkappa \text{id}_H & i\varkappa \text{id}_H \\ i\varkappa \text{id}_H & \bar{A}_2^t \Leftrightarrow i\varkappa \text{id}_H \end{pmatrix},$$

and we see immediately that $\mathcal{A}(\varkappa) \neq \bar{\mathcal{A}}^t(\varkappa)$ for $\varkappa > 0$, $\mathcal{A}(\varkappa)$ is not hermitian. To check whether $\mathcal{A}(\varkappa)$ is normal, calculate

$$\mathcal{A}(\varkappa)\bar{\mathcal{A}}^t(\varkappa) = \begin{pmatrix} A_1\bar{A}_1^t + i\varkappa(\bar{A}_1^t \Leftrightarrow A_1) + \varkappa^2 \text{id}_H & i\varkappa(A_1 \Leftrightarrow \bar{A}_2^t) \Leftrightarrow 2\varkappa^2 \text{id}_H \\ i\varkappa(\Leftrightarrow \bar{A}_1^t + A_2) \Leftrightarrow 2\varkappa^2 \text{id}_H & A_2\bar{A}_2^t + i\varkappa(\bar{A}_2^t \Leftrightarrow A_2) + \varkappa^2 \text{id}_H \end{pmatrix}$$

and

$$\bar{\mathcal{A}}^t(\varkappa)\mathcal{A}(\varkappa) = \begin{pmatrix} \bar{A}_1^t A_1 + i\varkappa(\bar{A}_1^t \Leftrightarrow A_2) + 2\varkappa^2 \text{id}_H & i\varkappa(\Leftrightarrow \bar{A}_1^t + A_2) \Leftrightarrow 2\varkappa^2 \text{id}_H \\ i\varkappa(A_1 \Leftrightarrow \bar{A}_2^t) \Leftrightarrow 2\varkappa^2 \text{id}_H & \bar{A}_2^t A_2 + i\varkappa(\bar{A}_2^t \Leftrightarrow A_2) + 2\varkappa^2 \text{id}_H \end{pmatrix}$$

so that $\mathcal{A}(\varkappa)$ is normal if and only if $A_1 = A_2$ and A_1 is normal.

IN A HILBERT SPACE OF INFINITE DIMENSION

For the case of infinite dimensional Hilbert spaces it has been noted that one might have different inner products $[\cdot, \cdot]_1$ and $[\cdot, \cdot]_2$ on the two copies H of the same space of vectors. Then, with regard to (2.28) with $\alpha = 0$, the adjoint of, say, $\begin{pmatrix} \text{id}_H & 0 \\ -i\varkappa R_1^{-i\varkappa} & \text{id}_H \end{pmatrix}$ of the form $\begin{pmatrix} \text{id}_H & B \\ 0 & \text{id}_H \end{pmatrix}$ is determined by the condition

$$\left\langle \begin{pmatrix} \text{id}_H & 0 \\ \Leftrightarrow i\varkappa R_1^{-i\varkappa} & \text{id}_H \end{pmatrix} \begin{pmatrix} U \\ V \end{pmatrix}, \begin{pmatrix} W \\ Z \end{pmatrix} \right\rangle = \left\langle \begin{pmatrix} U \\ V \end{pmatrix}, \begin{pmatrix} \text{id}_H & B \\ 0 & \text{id}_H \end{pmatrix} \begin{pmatrix} W \\ Z \end{pmatrix} \right\rangle$$

or, equivalently $[\Leftrightarrow i\varkappa R_1^{-i\varkappa} U, Z]_2 = [U, BZ]_1$, and for the adjoint of $\begin{pmatrix} \text{id}_H & -i\varkappa R_1^{-i\varkappa} \\ 0 & \text{id}_H \end{pmatrix}$ of the form $\begin{pmatrix} \text{id}_H & 0 \\ C & \text{id}_H \end{pmatrix}$ we require $[\Leftrightarrow i\varkappa R_1^{-i\varkappa} V, W]_1 = [U, CZ]_2$.

2.4.4 The resolvent of \mathcal{A}

From the Frobenius-Schur factorisation (2.28), at least formally, we get an expression of the resolvent $\mathcal{R}^\alpha = (\mathcal{A} \Leftrightarrow \alpha \text{id}_{\mathcal{H}})^{-1}$ in the form

$$\mathcal{R}^\alpha = \begin{pmatrix} \text{id}_H & i\varkappa R_1^{\alpha-i\varkappa} \\ 0 & \text{id}_H \end{pmatrix} \cdot \begin{pmatrix} R_1^{\alpha-i\varkappa} & 0 \\ 0 & [A_2 \Leftrightarrow (\alpha \Leftrightarrow i\varkappa) \text{id}_H + \varkappa^2 R_1^{\alpha-i\varkappa}]^{-1} \end{pmatrix} \begin{pmatrix} \text{id}_H & 0 \\ i\varkappa R_1^{\alpha-i\varkappa} & \text{id}_H \end{pmatrix}.$$

If the operator to the right exists then \mathcal{R}^α exists and α is a regular point of \mathcal{A} . Conversely, if $\alpha \in \sigma(\mathcal{A})$, then one of the operators $R_1^{\alpha-i\varkappa}$ or $T_1^{-1} := [A_2 \Leftrightarrow (\alpha \Leftrightarrow i\varkappa) \text{id}_H + \varkappa^2 R_1^{\alpha-i\varkappa}]^{-1}$ does not exist.

From (2.29) we get the expression

$$\mathcal{R}^\alpha = \begin{pmatrix} \text{id}_H & 0 \\ i\kappa R_2^{\alpha-i\kappa} & \text{id}_H \end{pmatrix} \cdot \begin{pmatrix} [A_1 \Leftrightarrow (\alpha \Leftrightarrow i\kappa)\text{id}_H + \kappa^2 R_2^{\alpha-i\kappa}]^{-1} & 0 \\ 0 & R_2^{\alpha-i\kappa} \end{pmatrix} \begin{pmatrix} \text{id}_H & i\kappa R_2^{\alpha-i\kappa} \\ 0 & \text{id}_H \end{pmatrix}.$$

Now, if $\alpha \in \sigma(\mathcal{A})$ then one of the operators $R_2^{\alpha-i\kappa}$ or $T_2^{-1} := [A_1 \Leftrightarrow (\alpha \Leftrightarrow i\kappa)\text{id}_H + \kappa^2 R_2^{\alpha-i\kappa}]^{-1}$ does not exist.

Chapter 3

A point mass in a random potential

In this chapter we study the question of stability in average for a simple example - the one-dimensional movement of a point mass in a potential which changes randomly. It is assumed to jump between two different potentials, one of which determines a stable, the other one an unstable dynamics, in a sense to be defined in the beginning of Section 3.1. The jumps shall be modelled by a symmetric continuous-time 2-state Markov process with exponentially distributed sojourn times with intensity \varkappa .

The example is simple in the way that the generator \mathcal{A} of the expectation semi-group is a 4×4 -matrix, the four eigenvalues can be calculated explicitly and sufficient conditions on the values of \varkappa and other characteristic parameters can be found analytically to obtain averaged stability for the model.

The description of the two dynamics with remarks to stability is done in Section 3.1. In Section 3.2 the eigenvalues of the operator \mathcal{A} are calculated and conditions are sought so that all these values have positive imaginary part to fulfil the conditions of Proposition A.30 for uniform boundedness of the expectation semi-group.

We will see that stability in average is possible if and only if the 'unstable' potential is somehow 'weaker' than the 'stable' one, and we will obtain some more detailed sufficient and necessary conditions.

In order to visualise the behaviour of the eigenvalues as functions of the intensity \varkappa , this dependence is illustrated graphically for different choices of parameters and values of \varkappa . This is described in Section 3.3.

3.1 Description of the model

Consider a point mass m sliding without friction along a line (with coordinates x) in a potential $V(x) = cx^2, c \in \mathbb{R}$. The movement of the point mass in time is then described by the differential equation

$$m\ddot{x}(t) = \Leftrightarrow \frac{d}{dx}V(x(t)) = \Leftrightarrow 2cx(t)$$

With some initial conditions and $\omega = \sqrt{\frac{2c}{m}} \in \mathbb{C}$ we obtain the initial value problem

$$\begin{cases} \Leftrightarrow \ddot{x}(t) = \omega^2 x(t) \\ x(0) = f, \dot{x}(0) = g. \end{cases} \quad (3.1)$$

the solution to this problem is

$$x(t) = f \cos(\omega t) + g \frac{\sin(\omega t)}{\omega}, \quad \omega \in \mathbb{C}.$$

On the two-dimensional space of Cauchy data

$$\mathcal{C} = \left\{ \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} : x_0 = x(t), x_1 = \dot{x}(t), x_0, x_1 \in \mathbb{C} \right\}$$

with standard basis we define as bilinear metric the **energy form** $[\cdot, \cdot]$ by

$$\begin{aligned} \left[\begin{pmatrix} x_0 \\ x_1 \end{pmatrix}, \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \right] &:= \frac{1}{2} (x_0 \ x_1) \begin{pmatrix} \omega^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \overline{y_0} \\ \overline{y_1} \end{pmatrix} \\ &= \frac{1}{2} (\omega^2 x_0 \overline{y_0} + x_1 \overline{y_1}) \end{aligned}$$

This definition is motivated by the fact that the energy in a physical sense¹ is conserved in time, since from (3.1)

$$\begin{aligned} 0 &= \ddot{x} + \omega^2 x \wedge \ddot{\bar{x}} + \omega^2 \bar{x} = 0 \\ \Rightarrow 0 &= \dot{x} (\ddot{x} + \omega^2 x) + \dot{\bar{x}} (\ddot{\bar{x}} + \omega^2 \bar{x}) \\ \Leftrightarrow 0 &= \frac{1}{2} \frac{d}{dt} (|\dot{x}|^2 + \omega^2 |x|^2) \\ \Leftrightarrow \frac{1}{2} (|\dot{x}|^2 + \omega^2 |x|^2) &= \text{const} \end{aligned} \quad (3.2)$$

On \mathcal{C} the initial value problem (3.1) is transformed into the Cauchy problem

$$\begin{cases} \frac{1}{i} \frac{d}{dt} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} = \begin{pmatrix} 0 & \Leftrightarrow i \\ i\omega^2 & 0 \end{pmatrix} \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \\ \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \Big|_{t=0} = \begin{pmatrix} f \\ g \end{pmatrix} \end{cases} \quad (3.3)$$

Define $A := \begin{pmatrix} 0 & -i \\ i\omega^2 & 0 \end{pmatrix}$, the generator of the evolution semi-group connected with (3.3). A is self-adjoint with respect to the energy form, since the following two terms are equal:

$$\begin{aligned} \left[A \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}, \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \right] &= \frac{1}{2} (\Leftrightarrow i x_1 \ i\omega^2 x_0) \begin{pmatrix} \omega^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \overline{y_0} \\ \overline{y_1} \end{pmatrix} \\ &= \frac{1}{2} (\Leftrightarrow i\omega^2 x_1 \overline{y_0} + i\omega^2 x_0 \overline{y_1}) \end{aligned}$$

and

$$\begin{aligned} \left[\begin{pmatrix} x_0 \\ x_1 \end{pmatrix}, A \begin{pmatrix} y_0 \\ y_1 \end{pmatrix} \right] &= \frac{1}{2} (x_0 \ x_1) \begin{pmatrix} \omega^2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \text{conj}(\Leftrightarrow i y_1) \\ \text{conj}(i\omega^2 y_0) \end{pmatrix} \\ &= i\omega^2 x_0 \overline{y_1} \Leftrightarrow i\omega^2 x_1 \overline{y_0} \end{aligned}$$

¹One can interpret the term in equation (3.2) as the sum of kinetic and potential energy.

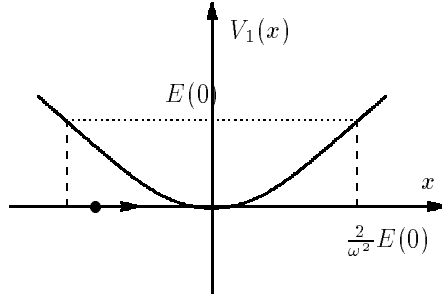


Figure 3.1: The point mass in the stable potential.

STABILITY

As definition of stability for this system we will take the natural one coming from the physical picture. We say that the evolution is stable if, for all times t , the position of the point mass is limited by a fixed finite interval and there exists a maximal velocity. In other words, the evolution is stable if there exist constants $M_1, M_2 > 0$ such that $|x(t)| \leq M_1$ and $|\dot{x}(t)| \leq M_2$ for all $t \geq 0$ (cf. Fig. 3.1).

Now let the potential $V(x)$ jump between $V_1(x)$ and $V_2(x)$, which determine a stable and an unstable dynamics respectively. For brevity call V_1 the stable and V_2 the unstable potential.

3.1.1 The stable dynamics

Let $V_1(x) = c_1 x^2$, $c_1 > 0$. Then $\omega_1 = \sqrt{\frac{2c_1}{m}}$ is real, the energy form

$$\left[\begin{pmatrix} x_0 \\ x_1 \end{pmatrix}, \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \right]_1 = \frac{1}{2} (\omega_1^2 |x_0|^2 + |x_1|^2)$$

is positive and defines a norm $\|\cdot\|_1$ on the space of Cauchy data $(\mathcal{C}_1, [\cdot, \cdot]_1)$. Since the energy norm of a vector is constant (cf. (3.2)) the evolution semi-group $T_1(t)$ generated by

$$A_1 := \begin{pmatrix} 0 & \Leftrightarrow i \\ i\omega_1^2 & 0 \end{pmatrix}$$

is unitary (A_1 is self-adjoint). Thus for the displacement and velocity we have

$$\begin{aligned} |x(t)|^2 &\leq \frac{1}{\omega_1^2} (\omega_1^2 |x_0|^2 + |x_1|^2) =: \frac{2}{\omega_1^2} E(0) & \forall t \geq 0 \\ \text{and } |\dot{x}(t)|^2 &\leq (\omega_1^2 |x_0|^2 + |x_1|^2) = 2E(0) & \forall t \geq 0, \end{aligned}$$

so that the evolution is stable, see Fig.3.1. This can also be seen from the fact that the solutions of the initial value problem (3.1) for $V = V_1$ are the trigonometric functions

$$x(t) = f \cos(\omega_1 t) + g \frac{\sin(\omega_1 t)}{\omega_1},$$

so that $|x(t)|$ and $|\dot{x}(t)|$ are bounded uniformly for all times t .

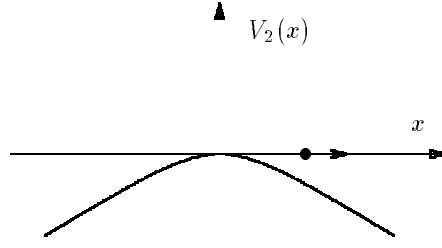


Figure 3.2: The point mass in the unstable potential.

3.1.2 The unstable dynamics

For $V_2(x) = \Leftrightarrow c_2 x^2, c_2 > 0$ the situation is quite different, here $\omega = i\sqrt{\frac{2c_2}{m}}$ is complex. To simplify notation define $\omega_2^2 := \frac{2c_2}{m} \Rightarrow \omega^2 = \Leftrightarrow \omega_2^2$.

The energy form

$$\left[\begin{pmatrix} x_0 \\ x_1 \end{pmatrix}, \begin{pmatrix} x_0 \\ x_1 \end{pmatrix} \right]_2 = \frac{1}{2} (\Leftrightarrow \omega_2^2 |x_0|^2 + |x_1|^2)$$

on the space \mathcal{C}_2 of Cauchy data is an indefinite metric (cf. Appendix A.5) with one-dimensional negative subspace.

The eigenvalues of the generator

$$A_2 := \begin{pmatrix} 0 & \Leftrightarrow i \\ \Leftrightarrow i \omega_2^2 & 0 \end{pmatrix}$$

(with respect to any norm) are $i\omega_2$ and $\Leftrightarrow i\omega_2$ with eigenvectors $(\frac{1}{i\omega_2}, 1)^t$ and $(\Leftrightarrow \frac{1}{i\omega_2}, 1)^t$, so that the eigenvalues of the evolution semi-group are $e^{-\omega_2 t}$ and $e^{\omega_2 t}$. There exists an exponentially increasing eigenvector, the point mass will approach to ∞ (see Fig. 3.2) - the dynamics is unstable. Again this can be easily seen from the solutions to (3.1) (with $\omega = i\omega_2$)

$$x(t) = f \cosh(\omega_2 t) + g \frac{\sinh(\omega_2 t)}{\omega_2}$$

Here we have hyperbolic functions, both $|x(t)|$ and $|\dot{x}(t)|$ tend to ∞ , unless of course $\begin{pmatrix} f \\ g \end{pmatrix}$ is the eigenvector to the eigenvalue $i\omega_2$.

3.1.3 The expectation semi-group

Now assume the random process is 'turned on', the potential $V(x)$ jumps between the two values $V_1(x)$ and $V_2(x)$ via a two-state continuous time Markov chain with infinitesimal generator

$$Q := \varkappa \begin{pmatrix} \Leftrightarrow 1 & 1 \\ 1 & \Leftrightarrow 1 \end{pmatrix}, \quad \varkappa > 0.$$

Then as described in Chapter 2 the generator \mathcal{A} of the expectation semi-group $\mathcal{E}(t) = e^{i\mathcal{A}t}$ acting on the space $\mathcal{C}_1 \times \mathcal{C}_2$ is

$$\mathcal{A}(\varkappa) = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} + i\varkappa \begin{pmatrix} \text{id}_H & \Leftrightarrow \text{id}_H \\ \Leftrightarrow \text{id}_H & \text{id}_H \end{pmatrix},$$

and since the operators A_1 and A_2 are represented w.r.t same basis (cf. Remark 2.23(1)), we get

$$\mathcal{A}(\varkappa) = \begin{pmatrix} i\varkappa & \Leftrightarrow i & \Leftrightarrow i\varkappa & 0 \\ i\omega_1^2 & i\varkappa & 0 & \Leftrightarrow i\varkappa \\ \Leftrightarrow i\varkappa & 0 & i\varkappa & \Leftrightarrow i \\ 0 & \Leftrightarrow i\varkappa & \Leftrightarrow i\omega_2^2 & i\varkappa \end{pmatrix}.$$

3.2 Conditions for stability in average

Note that here in finite dimensional space we need not worry about the norm we choose for the definition of stability, since all are equivalent. The expected evolution is stable if and only if all different eigenvalues of the generator \mathcal{A} have non-negative imaginary part, regardless of what norm we have chosen (cf. Proposition A.30). Depending on the parameters ω_1^2 and ω_2^2 we want to find values of \varkappa for which this is the case.

To emphasise the relation between the stable and unstable dynamics (and to simplify calculations) we define

$$r := \left| \frac{\omega_1^2}{\omega_2^2} \right| = \frac{c_1}{c_2}$$

For $r > 1$ the 'stable' potential $V_1(x)$ is a steeper parabola than for the 'unstable' one, and for $0 < r \leq 1$ the converse is true. (see Fig. 3.3).

Using the programme 'Maple', we calculate the four eigenvalues of $\mathcal{A}(\varkappa)$

$$\begin{aligned} \lambda_1(\varkappa) &= i\varkappa + \sqrt{\alpha + \frac{\omega_2}{2}\sqrt{\beta}}, & \lambda_2(\varkappa) &= i\varkappa \Leftrightarrow \sqrt{\alpha + \frac{\omega_2}{2}\sqrt{\beta}} \\ \lambda_3(\varkappa) &= i\varkappa + \sqrt{\alpha \Leftrightarrow \frac{\omega_2}{2}\sqrt{\beta}}, & \lambda_4(\varkappa) &= i\varkappa \Leftrightarrow \sqrt{\alpha \Leftrightarrow \frac{\omega_2}{2}\sqrt{\beta}} \end{aligned} \quad (3.4)$$

with

$$\alpha = \Leftrightarrow \varkappa^2 \Leftrightarrow \frac{1}{2}\omega_2^2(1 \Leftrightarrow r) \quad (3.5)$$

$$\beta = \omega_2^2(1+r)^2 + 8\varkappa^2(1 \Leftrightarrow r) \quad (3.6)$$

3.2.1 Stable potential 'weaker' than unstable potential

First let $0 < r \leq 1$. From (3.5) and (3.6) we see that in this case $\alpha \leq \Leftrightarrow \varkappa^2$ and $\beta > 0$. Then

$$\alpha \Leftrightarrow \frac{1}{2}\omega_2\sqrt{\beta} < \Leftrightarrow \varkappa^2 \Rightarrow \text{Im} \left(\sqrt{\alpha \Leftrightarrow \frac{1}{2}\omega_2\sqrt{\beta}} \right) > \varkappa.$$

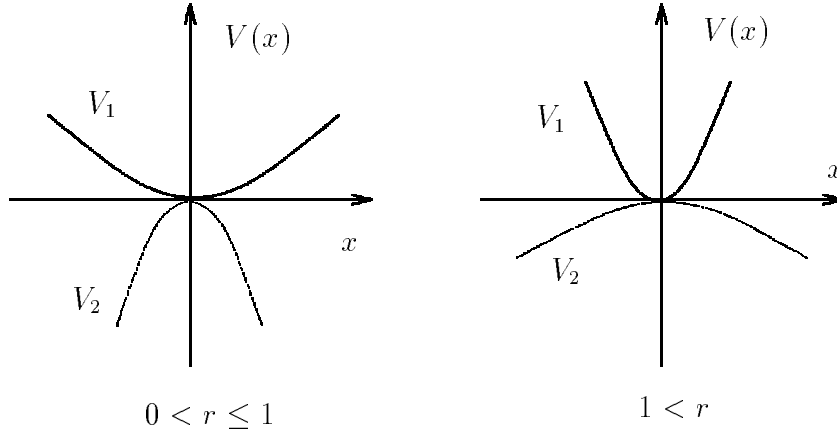


Figure 3.3: Comparison of the two potentials.

Hence for the imaginary part of λ_4 we get

$$\text{Im}(\lambda_4) = \varkappa \Leftrightarrow \text{Im} \left(\sqrt{\alpha \Leftrightarrow \frac{1}{2}\omega_2\sqrt{\beta}} \right) < 0 \quad \forall \varkappa > 0,$$

the generator $\mathcal{A}(\varkappa)$ has an eigenvalue in the LHP for all \varkappa , stability in average is not possible.

Thus a necessary condition for stability in average is

$$\omega_1^2 > \omega_2^2 \Leftrightarrow c_1 > c_2 \quad (N1)$$

3.2.2 Stable potential 'stronger' than unstable potential

Now let $r > 1$. We divide the calculations into two parts: the cases $\beta > 0$ and $\beta < 0$. The case $\beta = 0$ would require more detailed calculations of the geometric multiplicity of the eigenvalues, since then some eigenvalues are the same, but this shall not be of special interest here.

(1) Consider first the case when $\beta > 0$, so that $\sqrt{\beta} \in \mathbb{R}_+$. This holds for (see (3.6))

$$\beta > 0 \Leftrightarrow \varkappa < \sqrt{\frac{\omega_2^2(1+r)^2}{8(r \Leftrightarrow 1)}} =: K_2, \quad (3.7)$$

and then we get

$$\begin{aligned} \text{Im}(\lambda_4(\varkappa)) \geq 0 &\Leftrightarrow \alpha \Leftrightarrow \frac{1}{2}\omega_2\sqrt{\beta} \geq \Leftrightarrow \varkappa^2 \\ &\Leftrightarrow \frac{1}{2}\omega_2^2(r \Leftrightarrow 1) \geq \frac{1}{2}\omega_2\sqrt{\omega_2^2(1+r)^2 \Leftrightarrow 8\varkappa^2(r \Leftrightarrow 1)} \\ &\Leftrightarrow \varkappa^2 \geq \frac{\omega_2^2 r}{2(r \Leftrightarrow 1)}, \end{aligned}$$

that is

$$\text{Im}(\lambda_4(\varkappa)) \geq 0 \Leftrightarrow \varkappa \geq \sqrt{\frac{\omega_2^2 r}{2(r \Leftrightarrow 1)}} =: K_1. \quad (3.8)$$

Note that

$$K_1^2 = \frac{\omega_2^2 r}{2(r \Leftrightarrow 1)} < \frac{\omega_2^2 (1+r)^2}{8(r \Leftrightarrow 1)} = K_2^2$$

since

$$\frac{r}{2} < \frac{1}{8} (1 + 2r + r^2) \Leftrightarrow 0 < \frac{1}{8} (1 \Leftrightarrow r)^2$$

is true, and that for $\beta > 0$ from $Im(\lambda_4) \geq 0$ immediately follows for all other (different) eigenvalues that $Im(\lambda_l) \geq 0$ ($l = 1, 2, 3$) (cf. (3.4)). Thus we have found a first sufficient condition for stability in average

$$r > 1 \quad \wedge \quad K_1 \leq \varkappa < K_2 \quad (S1)$$

and the more detailed necessary condition

$$r > 1 \quad \wedge \quad \varkappa \geq K_1 \quad (N2)$$

The upper bound K_2 is motivated only by the constraint $\beta > 0$, so that, due to continuity in \varkappa , one can expect stability in average at least for a slightly bigger interval than $[K_1, K_2)$.

- (2) For $\varkappa > K_2$ it is $\beta < 0$, and $\alpha \pm \frac{\omega_2}{2} \sqrt{\beta}$ are complex numbers with imaginary parts $\pm \frac{\omega_2}{2} \sqrt{|\beta|}$. We write these numbers in polar form and define

$$\alpha + i \frac{\omega_2}{2} \sqrt{|\beta|} =: \mu^2 e^{i\Phi}, \quad (3.9)$$

$$\text{and thus } \alpha \Leftrightarrow i \frac{\omega_2}{2} \sqrt{|\beta|} = \mu^2 e^{-i\Phi},$$

with the argument $\Phi \in (0, \pi)$ and the absolute value μ^2 , where

$$\begin{aligned} \mu^4 &= \alpha^2 + \frac{\omega_2^2}{4} |\beta| \\ \Rightarrow \mu^4 &= \varkappa^4 + \varkappa^2 [\omega_2^2 (r \Leftrightarrow 1)] \Leftrightarrow \omega_2^4 r \end{aligned} \quad (3.10)$$

Note that $\mu^4 > 0$ because the roots of the polynomial in (3.10) are $\varkappa^2 = \omega_2^2$ and $\varkappa^2 = \Leftrightarrow \omega_2^2 r$, whereas $K_2^2 \geq \omega_2^2$ since (cf. (3.7))

$$\begin{aligned} 1 + 2r + r^2 &\geq 8(r \Leftrightarrow 1) \\ \Leftrightarrow (r \Leftrightarrow 3)^2 &\geq 0 \end{aligned}$$

and $\varkappa > K_2$ by assumption. Then we have (note the notation about the roots of complex numbers and see Fig. 3.4)

$$\pm \sqrt{\alpha + i \frac{\omega_2}{2} \sqrt{|\beta|}} = \pm \mu e^{i\frac{\Phi}{2}} \quad (3.11)$$

$$\pm \sqrt{\alpha \Leftrightarrow i \frac{\omega_2}{2} \sqrt{|\beta|}} = \mp \mu e^{-i\frac{\Phi}{2}} \quad (3.12)$$

And thus

$$\begin{aligned} Im(\lambda_1(\varkappa)) &= \varkappa + \mu \sin\left(\frac{\Phi}{2}\right), & Im(\lambda_2(\varkappa)) &= \varkappa \Leftrightarrow \mu \sin\left(\frac{\Phi}{2}\right) \\ Im(\lambda_3(\varkappa)) &= \varkappa + \mu \sin\left(\frac{\Phi}{2}\right), & Im(\lambda_4(\varkappa)) &= \varkappa \Leftrightarrow \mu \sin\left(\frac{\Phi}{2}\right) \end{aligned} \quad (3.13)$$

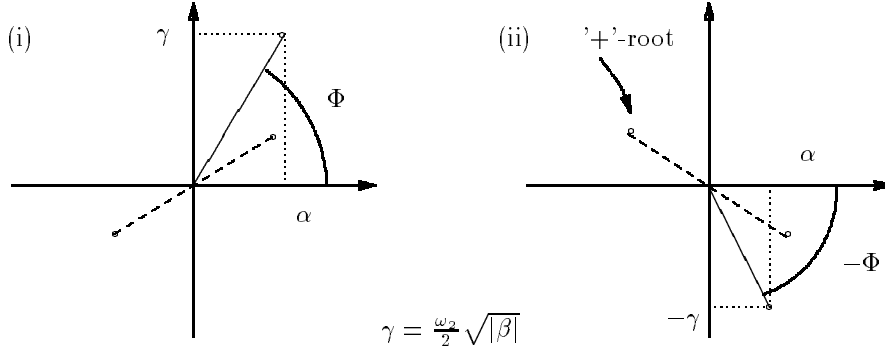


Figure 3.4: Illustration for equations (3.11) and (3.11).

Again we consider two cases:

- (a) As a first step we observe that, since $|\sin(\frac{\Phi}{2})| \leq 1$, all imaginary parts are certainly non-negative if $\mu \leq \varkappa$. That is with (3.10)

$$\begin{aligned} \mu \leq \varkappa &\Leftrightarrow \mu^4 \leq \varkappa^4 \Leftrightarrow \varkappa^4 + \varkappa^2 [\omega_2^2(r \Leftrightarrow 1)] \Leftrightarrow \omega_2^4 r \leq \varkappa^4 \\ &\Leftrightarrow \varkappa \leq \sqrt{\frac{\omega_2^2 r}{(r \Leftrightarrow 1)}} =: K_3 \end{aligned} \quad (3.14)$$

As we already have the sufficient condition (S1), $K_1 \leq \varkappa \leq K_2$, (3.14) gives a new condition only if $K_2 \leq K_3$ or with (3.7)

$$\begin{aligned} \frac{\omega_2^2(1+r)^2}{8(r \Leftrightarrow 1)} &\leq \frac{\omega_2^2 r}{(r \Leftrightarrow 1)} \\ \Leftrightarrow r^2 \Leftrightarrow 6r + 1 &\leq 0 \\ \Leftrightarrow 3 \Leftrightarrow 2\sqrt{2} &\leq r \leq 3 + 2\sqrt{2} \end{aligned}$$

With the constraint $r > 1$ we get that $K_2 \leq K_3$ if and only if $1 < r \leq 3 + 2\sqrt{2}$.

So far we have as sufficient conditions for stability

$$\begin{aligned} (1) \quad &1 < r \leq 3 + 2\sqrt{2} \quad \wedge \quad K_1 \leq \varkappa \leq K_3 \\ (2) \quad &3 + 2\sqrt{2} < r \quad \wedge \quad K_1 \leq \varkappa \leq K_2 \end{aligned}$$

- (b) The case $\mu > \varkappa$ is more complicated to deal with.

The calculation of the argument Φ in (3.9) can be split into two parts: $\alpha \geq 0$ and $\alpha < 0$. We have for $\alpha \geq 0$ (cf. Fig. 3.4)

$$\begin{aligned} \Phi &= \arctan\left(\frac{\omega_2 \sqrt{|\beta|}}{2\alpha}\right) \\ &= \arctan\left(\frac{\omega_2 \sqrt{8\varkappa^2(r \Leftrightarrow 1) \Leftrightarrow \omega_2^2(r+1)^2}}{\omega_2^2(r \Leftrightarrow 1) \Leftrightarrow 2\varkappa^2}\right) \in (0, \frac{\pi}{2}] \end{aligned} \quad (3.15)$$

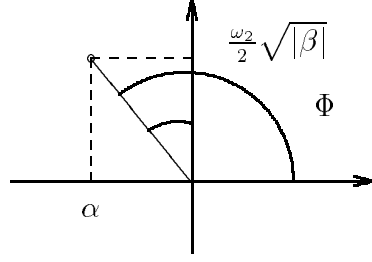


Figure 3.5: The argument Φ for $\alpha > 0$.

and for $\alpha < 0$ we get (cf. Fig. 3.5)

$$\begin{aligned}\Phi &= \frac{\pi}{2} + \arctan\left(\frac{2|\alpha|}{\omega_2\sqrt{|\beta|}}\right) \\ &= \frac{\pi}{2} + \arctan\left(\frac{2\kappa^2 \Leftrightarrow \omega_2^2(r \Leftrightarrow 1)}{\omega_2\sqrt{8\kappa^2(r \Leftrightarrow 1) \Leftrightarrow \omega_2^2(r+1)^2}}\right) \in \left(\frac{\pi}{2}, \pi\right)\end{aligned}\quad (3.16)$$

It is (cf. (3.5))

$$\alpha \geq 0 \Leftrightarrow \kappa \leq \sqrt{\frac{\omega_2^2}{2}(r \Leftrightarrow 1)} =: K_4. \quad (3.17)$$

For $1 < r \leq 3 + 2\sqrt{2}$ this case is only necessary studying when $K_3 < K_4$, i.e.

$$\begin{aligned}K_3 < K_4 &\Leftrightarrow \frac{\omega_2^2 r}{(r \Leftrightarrow 1)} < \frac{1}{2}\omega_2^2(r \Leftrightarrow 1) \\ &\Leftrightarrow r^2 \Leftrightarrow 4r + 1 > 0 \\ &\Leftrightarrow r > 2 + \sqrt{3} \vee r < 2 \Leftrightarrow \sqrt{3}.\end{aligned}$$

For $3 + 2\sqrt{2} < r$ we require $K_2 < K_4$, i.e.

$$\begin{aligned}K_2 < K_4 &\Leftrightarrow \frac{\omega_2^2(r+1)^2}{8(r \Leftrightarrow 1)} < \frac{1}{2}\omega_2^2(r \Leftrightarrow 1) \\ &\Leftrightarrow 3r^2 \Leftrightarrow 10r + 3 > 0 \\ &\Leftrightarrow r > 3 \vee r < \frac{2}{3}.\end{aligned}$$

With regard to (3.13) the requirement for stability is

$$\kappa \Leftrightarrow \mu \sin\left(\frac{\Phi}{2}\right) \geq 0 \Leftrightarrow \frac{\mu}{\kappa} \sin\left(\frac{\Phi}{2}\right) \leq 1.$$

For $\kappa > \max(K_3, K_4)$ we have $\mu/\kappa \geq 1$ with $\lim_{\kappa \rightarrow \infty} \mu/\kappa = 1$. However, we have also from (3.16)

$$\lim_{\kappa \rightarrow \infty} \frac{\Phi}{2} = \frac{\pi}{2}$$

thus $\lim_{\kappa \rightarrow \infty} \sin(\Phi/2) = 1$ as well. In order to find further sufficient conditions one will have to investigate the term $\mu/\kappa \sin(\Phi/2)$ in more detail; we will not do this here.

3.3 Summary and graphical illustration

So far the following cases have not been investigated:

$$\begin{array}{lll}
 (1) & 1 < r \leq 2 + \sqrt{3} & : K_3 < \varkappa \quad , \alpha < 0 \\
 (2) & 2 + \sqrt{3} < r \leq 3 + 2\sqrt{2} & : K_3 < \varkappa \leq K_4 \quad , \alpha \geq 0 \\
 & & K_4 < \varkappa \quad , \alpha < 0 \\
 (3) & 3 + 2\sqrt{2} < r & : K_2 < \varkappa \leq K_4 \quad , \alpha \geq 0 \\
 & & K_4 < \varkappa \quad , \alpha < 0
 \end{array}$$

and there has been found as

- sufficient conditions:

$$\begin{array}{ll}
 (1) & 1 < r \leq 3 + 2\sqrt{2} \quad \wedge \quad K_1 \leq \varkappa \leq K_3 \\
 (2) & 3 + 2\sqrt{2} < r \quad \wedge \quad K_1 \leq \varkappa \leq K_2
 \end{array}$$

- necessary condition:

$$r > 1 \quad \wedge \quad K_1 \leq \varkappa$$

with the constants

$$r = \frac{\omega_1^2}{\omega_2^2} \quad , K_1 = \sqrt{\frac{\omega_2^2 r}{2(r-1)}} \quad , K_2 = \sqrt{\frac{\omega_2^2(1+r)^2}{8(r-1)}} \quad , K_3 = \sqrt{\frac{\omega_2^2 r}{(r-1)}} \quad , K_4 = \sqrt{\frac{\omega_2^2(r-1)}{2}}$$

For various values of the parameters ω_1^2 and ω_2^2 we used the mathematical software 'Matlab' to calculate the eigenvalues for a sequence of choices of \varkappa and illustrate the results in graphs. A pseudo-code of the programme and some graphs for some values of ω_1^2 and ω_2^2 can be found in Appendix B.

The numbers behind 'omega₁²' and 'omega₂²' (see Figures B.1-5) are the values of the parameters ω_1^2 and ω_2^2 respectively which are valid for all graphs in that particular figure. The caption 'kappa=' at the y-axis show the values of \varkappa for the respective graph.

For $r > 1$ are values of the constants K_1, K_2, K_3 and K_4 printed on top of the second column. The eigenvalues with non-negative imaginary part are shown as 'o', whereas as an 'x' stands for an eigenvalue with negative imaginary part.

The values $\omega_2 = 1$ and $r = 0.8$ were chosen to produce the graphs in Fig. B.1. One could investigate further the reason that all eigenvalues for $\varkappa > 1$ apparently have zero real part (similarly for $\varkappa = 1.2$ and $\varkappa = 1.6$ in Fig. B.2). In accordance with the analytic results, there is always one eigenvalue with negative imaginary part, though this value seems to increase with increasing \varkappa .

The graphs in Fig. B.2 and B.3 visualise that for smaller values of r larger values of \varkappa are needed in order to 'lift' the eigenvalue in the LHP up into the UHP.

The sequence of graphs in Fig. B.4 with values of \varkappa up to 95 suggests that the averaged system is stable for large \varkappa , though it has not been shown analytically yet.

Note further that the picture of the four eigenvalues is symmetric to the imaginary

axis, one pair almost on the real line and the other pair with imaginary part approximately 2κ .

The conclusion we can draw from investigating the question of stability in average on the example in this chapter is summarised in

Hypothesis 3.1

- (1) *The necessary condition for stability in average for general systems with a 2-state Markov chains includes some condition on the relation between the parameters of the stable and unstable system, here it is the ratio r of the slopes of the potentials which needs to be larger than 1.*
- (2) *We can not expect the values of κ sufficient for stability to be small. This is important to know in order to apply methods of Perturbation Theory.*
- (3) *Since $\lim_{r \rightarrow 1+} K_1 = \infty$ in equation (3.8) it is even the case, that κ needs to be very large if the relation mentioned in (1) is almost 1.*

An interpretation of the result w.r.t. the Conjecture 1.1 is given in Chapter 6.

Chapter 4

The wave equation on the finite string with random boundary conditions

Here the question of stability in average is studied for the wave equation

$$Tu_{xx} = \varrho(x)u_{tt}$$

on the finite string ($x \in [0, N]$) with homogeneous tension T , continuous and positive density function $\varrho(x)$, and randomly changing boundary condition at $x = 0$.

We assume Dirichlet condition¹ $u|_N = 0$ at $x = N$, and at $x = 0$ the Rubin condition

$$u_x \Leftrightarrow h(t)u|_0 = 0, \quad h(t) \in \mathbb{R}.$$

Let $h(t)$ jump between the values $h_1 > 0$ and $h_2 < \Leftrightarrow \frac{1}{N}$, corresponding to a stable and unstable dynamics respectively, in a sense to be defined in the beginning of Section 4.1. Again the jump process shall be realised via a 2-state continuous-time Markov chain with exponentially distributed sojourn times with intensity \varkappa .

Since the main interest does not lie in the wave equation itself we simplify the calculations and study a 'solvable model' in which $T = 1$ and $\varrho(x) \equiv \rho^2 > 0$ are constant. However, many ideas employed and expressions obtained can easily be modified to suit the case of a general density function $0 < \gamma \leq \varrho(x) \leq \infty$ bounded below and above on $[0, N]$.

In Section 4.1 the analysis of the two different systems for $h = h_1 > 0$ and $h = h_2 < \Leftrightarrow \frac{1}{N}$ is given. First some facts and definitions are listed which hold for the general case. In the following subsections all the calculations of eigenvalues, eigenvectors, etc. are done as well as necessary definitions are given - first for the solvable model in general, then for both systems with $h = h_1$ and $h = h_2$ in particular.

Some of the relations between the domains, eigenvalues, etc., of the generators of the two dynamics are elaborated in Section 4.2. These relations are the basis for the further investigation of the properties of the generator \mathcal{A} of the expectation semi-group in Section 4.3, especially the spectral properties. Here we search for conditions on \varkappa

¹One distinguishes between Dirichlet ($u|_N = 0$), Neumann ($u'|_N = 0$) and Rubin condition ($au|_N + bu'|_N = 0$), see for example Strauss[33].

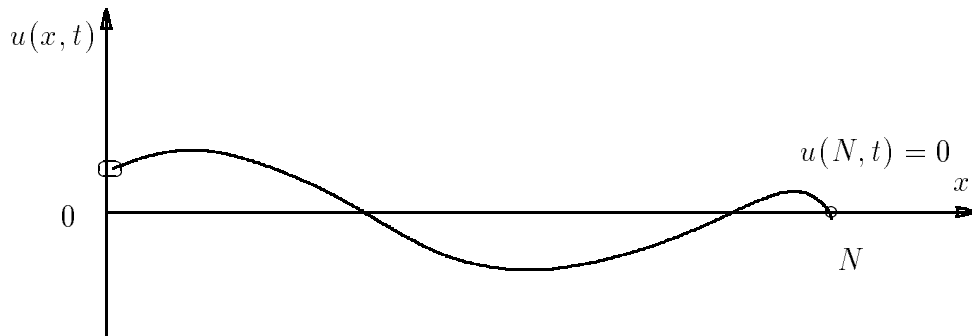


Figure 4.1: The finite string.

in dependence on the parameters h_1, h_2, T, N and ρ to obtain stability in average, i.e. an uniformly bounded expectation semi-group w.r.t. a particular norm. This norm is constructed in Section 4.1 by means of the canonical symmetry defined in the theory of spaces with indefinite metrics (see Appendix A.5), and it is shown that this norm is suitable to measure stability.

This situation is not as simple to deal with as the one-dimensional movement of a point mass described in the previous chapter. Here the operators involved act on a Hilbert space of infinite dimension, the generator \mathcal{A} of the expectation semi-group is of general type: unbounded in both directions and not symmetric. Additionally it is rather difficult to make use of the results of Perturbation Theory. The reason is the strong condition for stability, i.e. that the spectrum be completely contained in the complex UHP, together with the asymptotic behaviour of the spectrum of \mathcal{A} .

A numerical approach is realised in Section 4.4 to approximate the spectrum of \mathcal{A} . As described in Mikhlin[23], the Galerkin method provides a procedure to approximate the eigenvalues of a linear operator. Here we will use this method to visualise how the spectrum of \mathcal{A} looks like and how it changes in dependence on \varkappa and the other parameters.

The results of the numerical approach suggest that stability is in fact possible.

4.1 Description of the model

THE FINITE STRING

Consider a stretchable string of finite length N with density $\rho(x)$ fixed at the right end and a ring (idealised, i.e. without mass) attached to the left end. This ring shall slide along a rod in such a way that the slope of the string at $x = 0$ is proportional to its displacement. Also let the string be under tension T , so that any displacement from equilibrium results in an oscillation when released. Introduce a coordinate system (see Fig. 4.1) and let $u(x, t)$ be the displacement of the string at position x and time t .

STABILITY

Having in mind the model of the point mass in a potential, we shall mean that the dynamics of the string is stable if, released from its initial position $u(x, 0)$ with velocity $u_t(x, 0)$, the *energy term*

$$S(t) = \int_0^N |u_x|^2 + \varrho(x)|u_t|^2 dx,$$

is uniformly bounded at all times, that is if there exists a constant $M > 0$ such that $|S(t)| \leq M$ for all $t \geq 0$. Note that if $\int_0^N |u_x|^2 dx$ is bounded then the length of the string is bounded. This means that the displacement of the string is also bounded since the string is fixed at $x = N$, so that the oscillation is restricted to a bounded region.

The term $S(t)$ can be interpreted as the sum of kinetic and potential energy of the string (cf. Strauss[33]), in the following we will call it *physical energy* or simply *energy* of the string.

Unlike in the previous example, here we need to choose the norms on the considered spaces in such a way that stability in average follows from the fact that the expectation semi-group is equi-bounded. This will be done and justified in the Subsections 4.1.3 and 4.1.4 about the stable and unstable dynamics.

4.1.1 The case of a general density function

THE CAUCHY PROBLEM

The dynamics in time of the string (if we assume that any displacement is of small order compared with the total length N of the string) is described by the wave equation (cf. Strauss[33])

$$Tu_{xx} = \varrho(x)u_{tt} \tag{4.1}$$

with initial conditions

$$u(x, 0) = f(x) \quad , \quad u_t(x, 0) = g(x), \tag{4.2}$$

and some boundary conditions at $x = 0$ and $x = N$. Since the end at $x = N$ shall be fixed at all times we get as one condition $u|_N = u(N, t) = 0$ for all t . For the second condition at $x = 0$ the proportionality of slope and displacement translates to, with $h \in \mathbb{R}$,

$$u_x \Leftrightarrow hu \Big|_0 = 0 \quad \forall t \geq 0. \tag{4.3}$$

Introducing the Cauchy vector

$$U(t) = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} := \begin{pmatrix} u(x, t) \\ u_t(x, t) \end{pmatrix}$$

the wave equation (4.1) with initial conditions (4.2) can be written as the Cauchy problem

$$\begin{cases} \frac{1}{i} \frac{d}{dt} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = i \begin{pmatrix} 0 & \Leftrightarrow 1 \\ \Leftrightarrow \frac{T}{\varrho(x)} \frac{d^2}{dx^2} & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \\ \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \Big|_{t=0} = \begin{pmatrix} f \\ g \end{pmatrix} \end{cases} \tag{4.4}$$

where the boundary conditions are now encoded in the domains $\mathcal{D}(L)$ of the differential operator

$$L = \Leftrightarrow \frac{T}{\varrho(x)} \frac{d^2}{dx^2}$$

and in the domain $\mathcal{D}(A)$ of the infinitesimal generator

$$A = i \begin{pmatrix} 0 & \Leftrightarrow 1 \\ L & 0 \end{pmatrix}$$

of the evolution semi-group connected with (4.4).

Remark 4.1

Note for the following that for densities $0 < \gamma \leq \varrho(x) \leq \gamma < \infty$ bounded below and above the spaces $\mathcal{L}_{2,\varrho}[0, N]$ and $\mathcal{L}_{2,1/\varrho}[0, N]$ are isomorphic to $\mathcal{L}_2[0, N]$, the same holds for the corresponding Sobolev spaces. Thus in the following we will not distinguish between these spaces.

THE DOMAINS OF THE OPERATORS

For the domain of L it is easily seen that

$$\mathcal{D}(L) = \{y \in W_2^2 : y' \Leftrightarrow hy|_0 = 0, y|_N = 0\}.$$

The condition on the boundaries do make sense since y as well as y' are continuous. (see Theorem A.38(1)). We equip $\mathcal{D}(L)$ with the weighted \mathcal{L}_2 -inner product with weight function $\varrho(x)$, then L is a symmetric operator (see Appendix A.4). To obtain a suitable inner product for the domain of A observe that from (4.1) follows

$$\begin{aligned} 0 &\equiv Lu + u_{tt} \\ \Leftrightarrow 0 &= \langle Lu, u_t \rangle_\varrho + \langle u_{tt}, u_t \rangle_\varrho \\ \Leftrightarrow 0 &= \frac{1}{2} \frac{d}{dt} \left(\langle Lu, u \rangle_\varrho + \langle u_t, u_t \rangle_\varrho \right). \end{aligned}$$

That is, in the notation of Cauchy vectors, the following expression is conserved by the evolution

$$\begin{aligned} \frac{1}{2} \left(\langle Lu_0, u_0 \rangle_\varrho + \langle u_1, u_1 \rangle_\varrho \right) &= \frac{1}{2} \left(\int_0^N \Leftrightarrow \frac{d^2}{dx^2} u_0 \overline{u_0} + \varrho(x) |u_1|^2 dx \right) \\ &= \frac{1}{2} \left([\Leftrightarrow (u_0)_x \overline{u_0}]_0^N + \int_0^N |(u_0)_x|^2 + \varrho(x) |u_1|^2 dx \right), \\ \text{i.e. } \frac{1}{2} \left(\langle Lu_0, u_0 \rangle_\varrho + \langle u_1, u_1 \rangle_\varrho \right) &= \frac{1}{2} \left(h |u_0|^2|_0 + \int_0^N |(u_0)_x|^2 + \varrho(x) |u_1|^2 dx \right). \end{aligned} \quad (4.5)$$

We use the right hand side of equation (4.5) to define the **energy form** (which is not necessarily positive) for Cauchy vectors $U = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$ and $V = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix}$

$$\left[\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \right] := \frac{1}{2} \left(h u_0 \overline{v_0}|_0 + \int_0^N (u_0)_x \overline{(v_0)_x} + \varrho(x) u_1 \overline{v_1} dx \right), \quad (4.6)$$

and the space \mathcal{C}' of Cauchy data with finite energy

$$\mathcal{C}' := \left\{ \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} : u_0 \in W_2^1, u_1 \in \mathcal{L}_2 \right\}.$$

For the domain of A we get

$$\begin{aligned} \mathcal{D}(A) &= \left\{ U = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in \mathcal{C}' : u_0 \in \mathcal{D}(L), AU \in \mathcal{C}' \right\} \Rightarrow \\ &= \left\{ \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} : u_0 \in W_2^2, u_1 \in W_2^1, (u_0)_x \Leftrightarrow hu_0|_0 = 0, u_0|_N = 0, u_1|_N = 0 \right\}. \end{aligned}$$

The condition $u_1|_N = 0$ follows from the requirement that $u|_N \equiv 0$ since: $0 \equiv u_t|_N = u_1|_N$ follows from $u|_N \equiv 0$. We define the space

$$\mathcal{C} := \left\{ \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} : u_0 \in W_2^1, u_1 \in \mathcal{L}_2, u_0|_N = 0 \right\},$$

then A maps $\mathcal{D}(A)$ onto \mathcal{C} since

$$\begin{pmatrix} 0 & \Leftrightarrow i \\ iL & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} \Leftrightarrow i u_1 \\ iL u_0 \end{pmatrix} \in \mathcal{C},$$

and w.r.t. the energy form the closure of $\mathcal{D}(A)$ is \mathcal{C} (refer to Appendix C.1.1 for more details).

Remark 4.2

In order to use the left-hand side of equation (4.5) for calculating the energy form of two vectors $U, V \in \mathcal{C}$, it is only necessary that one vector be in $\mathcal{D}(A)$. Since then, w.l.o.g. let $U = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in \mathcal{D}(A)$, $V = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \in \mathcal{C}$ (note that $v_0|_N = 0$)

$$\begin{aligned} [U, V] &= \frac{1}{2} \left(hu_0 \overline{v_0}|_0 + \int_0^N (u_0)_x \overline{(v_0)_x} + \rho^2 u_1 \overline{v_1} dx \right) \\ &= \frac{1}{2} \left(hu_0 \overline{v_0}|_0 + [(u_0)_x \overline{v_0}]_0^N \Leftrightarrow \int_0^N (u_0)_{xx} \overline{v_0} + \rho^2 u_1 \overline{v_1} dx \right) \\ &= \frac{1}{2} \left(\int_0^N \rho^2 \left(\Leftrightarrow \frac{1}{\rho^2} \frac{d^2}{dx^2} u_0 \right) \overline{v_0} + \rho^2 u_1 \overline{v_1} dx \right) \\ &= \frac{1}{2} \left(\langle Lu_0, v_0 \rangle_{\rho^2} + \langle u_1, v_1 \rangle_{\rho^2} \right). \end{aligned}$$

A is symmetric with respect to the energy form, as for $U, V \in \mathcal{D}(A)$ we have

$$\begin{aligned} [AU, V] &= \frac{1}{2} \left(\langle L(\Leftrightarrow i u_1), v_0 \rangle_{\ell} + \langle iL u_0, v_1 \rangle_{\ell} \right) \\ &= \frac{1}{2} \left(\langle u_1, iL v_0 \rangle_{\ell} + \langle u_0, L(\Leftrightarrow i v_1) \rangle_{\ell} \right) = [U, AV] \end{aligned}$$

THE RESOLVENT OF A

Let $\alpha \in \mathbb{C}$ be in the resolvent set of A , and $R^\alpha = (A \Leftrightarrow \alpha \text{id}_{\mathcal{C}})^{-1}$ the resolvent of A .

For $\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in \mathcal{D}(A)$ and $\begin{pmatrix} f \\ g \end{pmatrix} \in \mathcal{C}$ we get

$$\begin{aligned} R^\alpha \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} &\Leftrightarrow \begin{pmatrix} \Leftrightarrow \alpha & \Leftrightarrow i \\ iL & \Leftrightarrow \alpha \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \\ &\Leftrightarrow \begin{cases} u_1 = if + i\alpha u_0 \\ (L \Leftrightarrow \alpha^2)u_0 = (\Leftrightarrow ig + \alpha f) \end{cases} \end{aligned} \quad (4.7)$$

So to find $\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}$ we need to solve the boundary value problem

$$\begin{cases} (L \Leftrightarrow \alpha^2)y = (\Leftrightarrow ig + \alpha f) \\ y_x \Leftrightarrow hy|_0 = 0 \\ y|_N = 0 \end{cases}$$

This is done with the help of the Green's function $G^\alpha(x, \xi)$ (refer to Appendix A.4). We get for u_0 in (4.7)

$$u_0 = \int_0^N G^\alpha(x, \xi) \varrho(\xi) [\Leftrightarrow ig(\xi) + \alpha f(\xi)] d\xi,$$

or with the corresponding integral operator K^α of Hilbert-Schmidt type we have

$$u_0 = K^\alpha(\Leftrightarrow ig + \alpha f) \quad \wedge \quad u_1 = if + i\alpha K^\alpha(\Leftrightarrow ig + \alpha f).$$

Finally we obtain an expression for the resolvent R^α

$$R^\alpha \begin{pmatrix} f \\ g \end{pmatrix} = [A \Leftrightarrow \alpha \text{id}_{\mathcal{C}}]^{-1} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \alpha K^\alpha & \Leftrightarrow i K^\alpha \\ i + i\alpha^2 K^\alpha & \alpha K^\alpha \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} \quad (4.8)$$

SELF-ADJOINTNESS

If L is positive the energy form $[\cdot, \cdot]$ in (4.6) is positive, and A is even self-adjoint since in that case for $(A \Leftrightarrow i \text{id}_{\mathcal{C}})$ and $(A + i \text{id}_{\mathcal{C}})$ hold

$$\begin{aligned} (A \pm i \text{id}_{\mathcal{C}}) \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} &\Leftrightarrow \begin{cases} \pm i u_0 \Leftrightarrow i u_1 = f \\ i L u_0 \pm i u_1 = g \end{cases} \\ &\Leftrightarrow \begin{cases} u_1 = if \pm u_0 \\ (L + 1)u_0 = \Leftrightarrow ig \mp if \end{cases} \end{aligned}$$

As for positive L the resolvent $(L + 1)^{-1}$ exists and is defined on the whole of \mathcal{L}_2 , the resolvents $(A \pm i \text{id}_{\mathcal{C}})^{-1}$ exist on the whole of \mathcal{C} , the symmetric operator A is self-adjoint with Theorem A.4.

THE SPECTRUM

The spectra of A and L are closely connected. E.g. let λ be an eigenvalue of A ,

then there exist $U = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in \mathcal{D}(A)$ such that

$$\begin{aligned} A \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = \lambda \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} &\Leftrightarrow \begin{cases} \Leftrightarrow i u_1 = \lambda u_0 \\ i L u_0 = \lambda u_1 \end{cases} \\ &\Leftrightarrow \begin{cases} u_1 = i \lambda u_0 \\ L u_0 = \lambda^2 u_0 \end{cases} \end{aligned}$$

i.e. λ^2 is an eigenvalue of L . Conversely, simple calculations show that if λ is an eigenvalue of L (positive or negative) with eigenfunction u_λ then $\sqrt{\lambda}$ and $\Leftrightarrow\sqrt{\lambda}$ are eigenvalues of A with eigenvectors $\begin{pmatrix} \frac{1}{i\sqrt{\lambda}}u_\lambda \\ u_\lambda \end{pmatrix}$ and $\begin{pmatrix} -\frac{1}{i\sqrt{\lambda}}u_\lambda \\ u_\lambda \end{pmatrix}$ respectively. Since L is symmetric its eigenvalues are real, thus the eigenvalues of A are either real or purely imaginary, and further, the spectrum of A is symmetric to the real and to the imaginary axis. Also, the spectrum $\sigma(A)$ of A is discrete, due to the fact that the spectrum $\sigma(L)$ of L is discrete. With Remark A.14 follows now, that the system of eigenvectors forms a complete orthogonal set for \mathcal{C} .

4.1.2 A solvable model

From now on we simplify the calculations and consider a 'solvable model' by setting $T = 1$ and $\varrho(x) \equiv \rho^2 > 0$, we obtain the Cauchy problem

$$\begin{cases} \frac{1}{i} \frac{d}{dt} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} = i \begin{pmatrix} 0 & \Leftrightarrow 1 \\ \Leftrightarrow \frac{1}{\rho^2} \frac{d^2}{dx^2} & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \\ \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \Big|_{t=0} = \begin{pmatrix} f \\ g \end{pmatrix} \end{cases} \quad (4.9)$$

EIGENVALUES AND EIGENVECTORS

We saw that in order to find the eigenvalues $\pm\lambda$ of A , we need to find the (real) eigenvalues λ^2 of the Sturm-Liouville problem

$$\Leftrightarrow \frac{1}{\rho^2} \frac{d^2}{dx^2} y = \lambda^2 y \quad (4.10)$$

$$\left. \begin{array}{l} y' \Leftrightarrow h y|_0 = 0 \\ y|_N = 0 \end{array} \right\} \quad (4.11)$$

Consider the three cases:

(1) $\lambda^2 = 0$

A general solution for the differential equation (4.10) is

$$y = \alpha x + \beta,$$

the boundary conditions (4.11) give

$$\beta = \frac{1}{h} \alpha \wedge \beta = \Leftrightarrow N \alpha.$$

Thus $\lambda = 0$ is an eigenvalue of L (and then of A) if and only if $h = \Leftrightarrow \frac{1}{N}$. We will not investigate this special case further.

- (2) $\lambda^2 > 0, \lambda > 0$
 A general solution to (4.10) is

$$u_\lambda = \alpha \cos(\lambda \rho x) + \beta \sin(\lambda \rho x),$$

(4.11) gives

$$\beta = \frac{h}{\lambda \rho} \alpha \quad \wedge \quad \beta = \Leftrightarrow \alpha \cot(\lambda \rho N).$$

Thus λ^2 is an eigenvalue of L if and only if $\lambda > 0$ solves

$$\tan(\lambda \rho N) = \Leftrightarrow \frac{\lambda \rho}{h}. \quad (4.12)$$

Since $\tan(x)$ is periodic equation (4.12) has countable infinitely many solutions $\lambda_s, s \in \mathbb{N}$, for all $h \in \mathbb{R}$. The eigenvectors of A for the eigenvalues λ_s and $\lambda_{-s} = \Leftrightarrow \lambda_s$ are

$$\begin{pmatrix} \frac{1}{i\lambda_s} u_s \\ u_s \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \Leftrightarrow \frac{1}{i\lambda_s} u_s \\ u_s \end{pmatrix},$$

with

$$u_s(x) = \cos(\lambda_s \rho x) + \frac{h}{\lambda_s \rho} \sin(\lambda_s \rho x), \quad x \in [0, N]. \quad (4.13)$$

For the energy forms of the eigenvectors we get

$$\left[\begin{pmatrix} \pm \frac{1}{i\lambda_s} u_s \\ u_s \end{pmatrix}, \begin{pmatrix} \pm \frac{1}{i\lambda_s} u_s \\ u_s \end{pmatrix} \right] = \frac{1}{2} \left(\frac{1}{\lambda_s^2} \langle L u_s, u_s \rangle_\ell + \langle u_s, u_s \rangle_\ell \right) = \langle u_s, u_s \rangle_\ell, \quad (4.14)$$

and the eigenvectors are orthogonal to each other, since for $u_s \neq u_t$ we have

$$\begin{aligned} \left[\begin{pmatrix} \frac{1}{i\lambda_s} u_s \\ u_s \end{pmatrix}, \begin{pmatrix} \frac{1}{i\lambda_t} u_t \\ u_t \end{pmatrix} \right] &= \frac{1}{2} \left(\frac{1}{\lambda_s \lambda_t} \langle L u_s, u_t \rangle_\ell + \langle u_s, u_t \rangle_\ell \right) \\ &= \frac{1}{2} \left(\frac{\lambda_s}{\lambda_t} + 1 \right) \langle u_s, u_t \rangle_\ell = 0 \end{aligned}$$

Note that the eigenfunctions of L are orthogonal in $\mathcal{L}_{2,\ell}$ (cf. Appendix A.4).

- (3) $\lambda^2 = \Leftrightarrow l^2 < 0, l > 0$
 A general solution to (4.10) is

$$u(x) = \alpha \cosh(l \rho x) + \beta \sinh(l \rho x),$$

and (4.11) gives

$$\beta = \frac{h}{l \rho} \alpha \quad \wedge \quad \beta = \Leftrightarrow \alpha \coth(l \rho N).$$

Thus $\Leftrightarrow l^2$ is an eigenvalue of L if and only if $l > 0$ solves

$$\tanh(l \rho N) = \Leftrightarrow \frac{l \rho}{h} \quad (4.15)$$

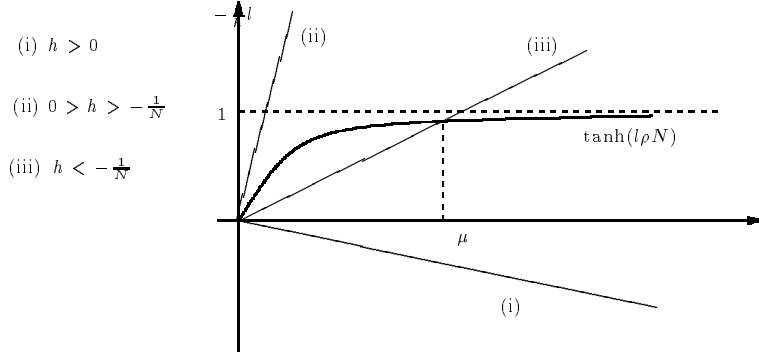


Figure 4.2: The negative eigenvalue of L .

The graphs of the functions in equation (4.15) are sketched in Fig. 4.2. Since

$$\frac{d}{dl} \tanh(l\rho N) \Big|_{l=0} = \rho N$$

there does not exist a solution to (4.15) for $h > \Leftrightarrow \frac{1}{N}$, then L is positive definite and A is self-adjoint. But for $h < \Leftrightarrow \frac{1}{N}$ there exists exactly one solution $\mu > 0$, L has one negative eigenvalue with eigenfunction

$$e_\mu(x) = \cosh(\mu\rho x) + \frac{h}{\mu\rho} \sinh(\mu\rho x), \quad x \in [0, N]. \quad (4.16)$$

and $i\mu$ and $\Leftrightarrow i\mu$ are eigenvalues of A with eigenvectors

$$\begin{pmatrix} \Leftrightarrow \frac{1}{\mu} e_\mu \\ e_\mu \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{1}{\mu} e_\mu \\ e_\mu \end{pmatrix}$$

respectively. For their energy forms we have here (in accordance with Proposition A.55)

$$\left[\begin{pmatrix} \pm \frac{1}{\mu} e_\mu \\ e_\mu \end{pmatrix}, \begin{pmatrix} \pm \frac{1}{\mu} e_\mu \\ e_\mu \end{pmatrix} \right] = \frac{1}{2} \left(\frac{1}{\mu^2} \langle L e_\mu, e_\mu \rangle_\varrho + \langle e_\mu, e_\mu \rangle_\varrho \right) = 0,$$

whereas

$$\left[\begin{pmatrix} \Leftrightarrow \frac{1}{\mu} e_\mu \\ e_\mu \end{pmatrix}, \begin{pmatrix} \frac{1}{\mu} e_\mu \\ e_\mu \end{pmatrix} \right] = \frac{1}{2} \left(\Leftrightarrow \frac{1}{\mu^2} \langle L e_\mu, e_\mu \rangle_\varrho + \langle e_\mu, e_\mu \rangle_\varrho \right) = \langle e_\mu, e_\mu \rangle_\varrho > 0.$$

However, the above eigenvectors are orthogonal to all of the eigenvectors for real eigenvalues since $\langle e_\mu, u_s \rangle_\varrho = 0$ and

$$\left[\begin{pmatrix} \pm \frac{1}{\mu} e_\mu \\ e_\mu \end{pmatrix}, \begin{pmatrix} \pm \frac{1}{i\lambda_s} u_s \\ u_s \end{pmatrix} \right] = \frac{1}{2} \left(\langle \pm \frac{1}{\mu} (\Leftrightarrow \mu^2) e_\mu, \pm \frac{1}{i\lambda_s} u_s \rangle_\varrho + \langle e_\mu, u_s \rangle_\varrho \right) = 0$$

To calculate the Green's function needed for the resolvent we follow the steps of the calculation of the resolvent $(L \Leftrightarrow \lambda)^{-1}$ in the general case done in Appendix A.4. The functions $l(x)$ and $r(x)$ are now

$$l(x) = \cos(\alpha\rho x) + \frac{h}{\alpha\rho} \sin(\alpha\rho x) \quad (4.17)$$

$$r(x) = \cos(\alpha\rho x) \Leftrightarrow \cot(\alpha\rho N) \sin(\alpha\rho x) \quad (4.18)$$

with Wronskian

$$W(l, r)|_{x=0} = \begin{vmatrix} 1 & 1 \\ h & \Leftrightarrow \alpha\rho \cot(\alpha\rho N) \end{vmatrix} = \Leftrightarrow h \Leftrightarrow \alpha\rho \cot(\alpha\rho N) = \Delta. \quad (4.19)$$

Then the Green's function is

$$G^\alpha(x, \xi) = \begin{cases} \frac{1}{h + \alpha\rho \cot(\alpha\rho N)} l(x)r(\xi) & , 0 \leq x \leq \xi \leq N \\ \frac{1}{h + \alpha\rho \cot(\alpha\rho N)} l(\xi)r(x) & , 0 \leq \xi \leq x \leq N \end{cases} \quad (4.20)$$

4.1.3 The stable system

As it will be seen in the following, the dynamics determined by the Cauchy problem (4.9) is stable when the value of h in the boundary condition at $x = 0$ in (4.3) is positive.² So let $h = h_1 > 0$, and define the differential operator $L_1 = \Leftrightarrow \frac{1}{\rho^2} \frac{d^2}{dx^2}$ with domain

$$\mathcal{D}(L_1) = \{y \in W_2^2 : y' \Leftrightarrow h_1 y|_0 = 0, y|_N = 0\},$$

and the infinitesimal generator $A_1 = i \begin{pmatrix} 0 & -1 \\ L_1 & 0 \end{pmatrix}$ with domain

$$\mathcal{D}(A_1) = \left\{ \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} : u_0 \in W_2^2, u_1 \in W_2^1, (u_0)_x \Leftrightarrow h_1 u_0|_0 = 0, u_0|_N = 0, u_1|_N = 0 \right\}.$$

THE ENERGY FORM AND STABILITY

The energy form $[\cdot, \cdot]_1$ (cf. (4.6))

$$\left[\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \right]_1 := \frac{1}{2} \left(h_1 u_0 \overline{v_0} + \int_0^N (u_0)_x \overline{(v_0)_x} + \rho^2 u_1 \overline{v_1} dx \right) \quad (4.21)$$

is positive and defines a norm $\|\cdot\|_1$ on the space

$$\mathcal{C}_1 := \left\{ \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} : u_0 \in W_2^1, u_1 \in \mathcal{L}_2, u_0|_N = 0 \right\}$$

of Cauchy data with finite energy norm. It follows immediately that the norm $\|\cdot\|_1$ is suitable to measure stability. Whenever an evolution semi-group is uniformly bounded

²This actually is true for $h_1 > -\frac{1}{N}$. However, we choose to treat only the case $h_1 > 0$, because although that case is qualitatively similar to the case $-\frac{1}{N} < h_1 < 0$ the calculations are quite different, and, besides, we can always decrease the distance of $h_1 > 0$ to the 'critical' value $-\frac{1}{N}$ by increasing the value of N .

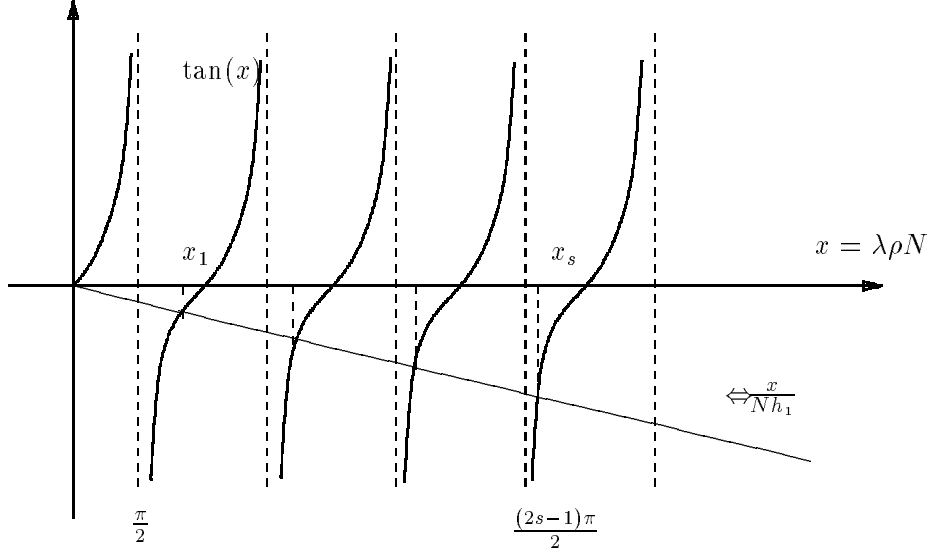


Figure 4.3: Approximation of the eigenvalues of A_1 .

w.r.t. the norm $\|\cdot\|_1$ then the physical energy

$$S(t) = \int_0^N |(u_0)_x|^2 + \rho^2 |u_1|^2 dx \leq \left[\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \right]_1$$

of the string is certainly uniformly bounded for $t \geq 0$ since $h_1 > 0$, in fact $e^{iA_1 t}$ is unitary, since A_1 is self-adjoint. Thus $h = h_1$ determines a stable dynamics.

THE EIGENVALUES AND EIGENVECTORS OF A_1

All the calculations have been done before, so here we only give the necessary definitions and certain estimations.

Observing the graphs of $\tan(\lambda\rho N)$ and $\Leftrightarrow \frac{\lambda\rho}{h_1}$ in the variable $\lambda\rho N$ (see Fig. 4.3) we get an approximation and the asymptotic behaviour of the positive solutions $\lambda_s, s \in \mathbb{N}$, of

$$\tan(\lambda\rho N) = \Leftrightarrow \frac{\lambda\rho}{h_1}. \quad (4.22)$$

We get for the eigenvalues λ_s and $\lambda_{-s} = \Leftrightarrow \lambda_s$

$$\lambda_s \rho N = \frac{2s-1}{2}\pi + \varepsilon_s \quad \wedge \quad \lambda_{-s} \rho N = \Leftrightarrow \frac{2s-1}{2}\pi \Leftrightarrow \varepsilon_s \quad (4.23)$$

with

$$\varepsilon_s \in (0, \frac{\pi}{2}), s \in \mathbb{N}, \quad \wedge \quad \lim_{s \rightarrow \infty} \varepsilon_s = 0,$$

and

$$\tan(\lambda_s \rho N) = \tan\left(\frac{\pi}{2} + \varepsilon_s\right) = \Leftrightarrow \cot(\varepsilon_s) = \Leftrightarrow \frac{1}{Nh_1} \left(\frac{2s-1}{2}\pi + \varepsilon_s \right) = \Leftrightarrow \frac{\lambda_s \rho}{h_1}.$$

An estimation for ε_s is

$$0 < \varepsilon_s \leq \tan(\varepsilon_s) = \frac{Nh_1}{\frac{2s-1}{2}\pi + \varepsilon_s} \leq \frac{2Nh_1}{(2s \Leftrightarrow 1)\pi} \leq \frac{Nh_1}{(s \Leftrightarrow 1)\pi}, \quad (4.24)$$

and for $|\lambda_s|$

$$|\lambda_s| = \frac{1}{N\rho} \left(\frac{2|s| \Leftrightarrow 1}{2} \pi + \varepsilon_s \right) \begin{cases} \leq \frac{|s|\pi}{N\rho} \\ \geq \frac{(2|s|-1)\pi}{2N\rho} \geq \frac{(s-1)\pi}{N\rho} \end{cases} \quad (4.25)$$

For the eigenvalues λ_s and λ_{-s} the eigenvectors of A_1 are

$$U_s := \begin{pmatrix} \frac{1}{i\lambda_s} u_s \\ u_s \end{pmatrix} \quad \text{and} \quad U_{-s} := \begin{pmatrix} \frac{1}{i\lambda_{-s}} u_s \\ u_s \end{pmatrix}, \quad s \in \mathbb{N}$$

resp. with $u_s(x) := \cos(\lambda_s \rho x) + \frac{h_1}{\lambda_s \rho} \sin(\lambda_s \rho x)$.

THE NORMS OF THE EIGENVECTORS

For the energy norms of the eigenvectors $U_{\pm s}$, $s \in \mathbb{Z}^*$ we get (for the calculations refer to Appendix C.1.2)

$$\|U_s\|_1^2 = \|U_{-s}\|_1^2 = \frac{\rho^2 N}{2} + h_1 \frac{Nh_1 + 1}{2\lambda_s^2} \geq \frac{\rho^2 N}{2} \quad (4.26)$$

Note that the norms of $\|U_{\pm s}\|_1^2$ are monotonically decreasing with increasing $s > 0$.

THE RESOLVENT OF A_1

From (4.17) and (4.18) we get for $\alpha \in \mathbb{C}$ not an eigenvalue of A_1

$$\begin{aligned} l_1(x) &= \cos(\alpha \rho x) + \frac{h_1}{\alpha \rho} \sin(\alpha \rho x) \\ r(x) &= \cos(\alpha \rho x) \Leftrightarrow \cot(\alpha \rho N) \sin(\alpha \rho x). \end{aligned}$$

Then we have from (4.19) and (4.20)

$$\begin{aligned} G_1^\alpha(x, \xi) &= \begin{cases} \frac{1}{h_1 + \alpha \rho \cot(\alpha \rho N)} l_1(x) r(\xi) & , 0 \leq x \leq \xi \leq N \\ \frac{1}{h_1 + \alpha \rho \cot(\alpha \rho N)} l_1(\xi) r(x) & , 0 \leq \xi \leq x \leq N \end{cases}, \\ K_1^\alpha &: y(\cdot) \mapsto \rho^2 \int_0^N G_1^\alpha(\cdot, \xi) y(\xi) d\xi, \end{aligned} \quad (4.27)$$

and from (4.8) the resolvent of A_1 defined on \mathcal{C}_1

$$R_1^\alpha \begin{pmatrix} f \\ g \end{pmatrix} = [A_1 \Leftrightarrow \alpha \text{id}_{\mathcal{C}_1}]^{-1} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \alpha K_2^\alpha & \Leftrightarrow i K_2^\alpha \\ i + i\alpha^2 K_2^\alpha & \alpha K_2^\alpha \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}. \quad (4.28)$$

4.1.4 The unstable system

The situation is quite different when $h = h_2 < \Leftrightarrow \frac{1}{N}$. Then the differential operator $L_2 = \Leftrightarrow \frac{1}{\rho^2} \frac{d^2}{dx^2}$ with domain

$$\mathcal{D}(L_2) = \{y \in W_2^2 : y' \Leftrightarrow h_2 y|_0 = 0, y|_N = 0\}$$

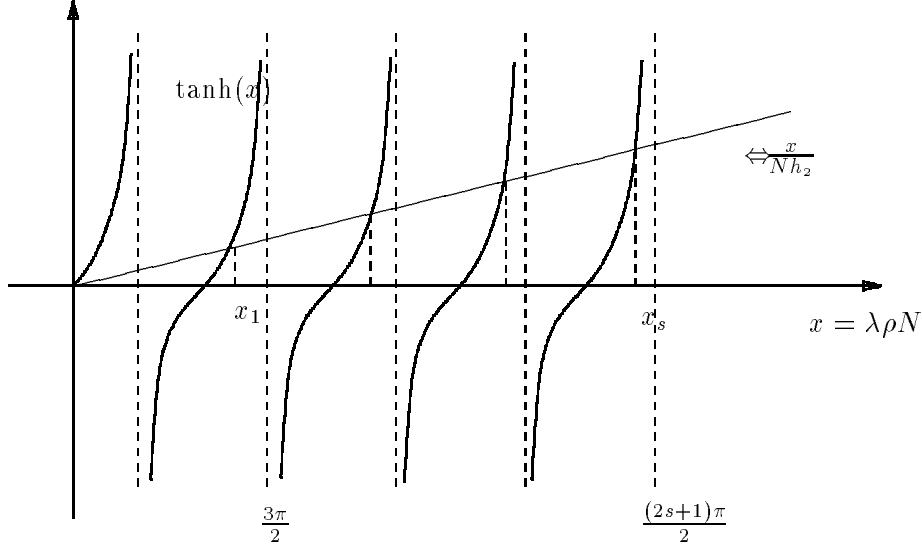


Figure 4.4: Approximation if the eigenvalues of A_2 .

has got one negative eigenvalue $\Leftrightarrow \mu^2$ (cf. Section 4.1.2.). The generator $A_2 = i \begin{pmatrix} 0 & -1 \\ L_2 & 0 \end{pmatrix}$ with domain

$$\mathcal{D}(A_2) = \left\{ \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} : u_0 \in W_2^2, u_1 \in W_2^1, (u_0)_x \Leftrightarrow h_2 u_0|_0 = 0, u_0|_N = 0, u_1|_N = 0 \right\}$$

of the evolution semi-group $e^{iA_2 t}$ corresponding to the Cauchy problem (4.9) has got the two complex eigenvalues $i\mu$ and $\Leftrightarrow i\mu$, so that $e^{iA_2 t}$ itself has got the eigenvalues $e^{-\mu t}$ and $e^{\mu t}$. Thus there exists an exponentially increasing eigenvector, the physical energy $S(t)$ of this particular state will approach to infinity in time - the system is unstable.

THE EIGENVALUES AND EIGENVECTORS OF A_2

- (1) The real eigenvalues $\tau_s > 0$ and $\tau_{-s} = \Leftrightarrow \tau_s$ ($s \in \mathbb{N}$) of A_2 are given by the positive solutions of

$$\tan(\tau \rho N) = \Leftrightarrow \frac{\tau \rho}{h_2}. \quad (4.29)$$

Again observing the graphs of the functions sketched in Fig. 4.4, and with

$$\frac{d}{d\tau} \tan(\tau \rho N) \Big|_{\tau=0} = \rho N > \Leftrightarrow \frac{\rho}{h_2} = \frac{d}{d\tau} \left(\Leftrightarrow \frac{\tau \rho}{h_2} \right) \Big|_{\tau=0}$$

we get as an approximation for $\tau_{\pm s}$

$$\tau_s \rho N = \frac{2s+1}{2} \pi \Leftrightarrow \delta_s \quad \wedge \quad \tau_{-s} \rho N = \Leftrightarrow \frac{2s+1}{2} \pi + \delta_s \quad (4.30)$$

with

$$\delta_s \in (0, \frac{\pi}{2}), s \in \mathbb{N}, \quad \wedge \quad \lim_{s \rightarrow \infty} \delta_s = 0,$$

and also

$$\tan(\tau_s \rho N) = \tan\left(\frac{\pi}{2} \Leftrightarrow \delta_s\right) = \cot(\delta_s) = \Leftrightarrow \frac{1}{N h_2} \left(\frac{2s+1}{2} \pi \Leftrightarrow \delta_s \right) = \Leftrightarrow \frac{\tau_s \rho}{h_2}.$$

Hence we can estimate δ_s by

$$0 < \delta_s \leq \tan(\delta_s) = \Leftrightarrow \frac{N h_2}{\frac{2s+1}{2} \pi \Leftrightarrow \delta_s} \leq \Leftrightarrow \frac{N h_2}{s \pi}, \quad (4.31)$$

and $|\tau_{\pm s}|$

$$|\tau_{\pm s}| = \frac{1}{N \rho} \left(\frac{2|s|+1}{2} \pi \Leftrightarrow \delta_{|s|} \right) \begin{cases} \leq & \frac{(2|s|+1)\pi}{2N\rho} \leq \frac{(|s|+1)\pi}{N\rho} \\ \geq & \frac{|s|\pi}{N\rho} \end{cases} \quad (4.32)$$

The eigenvectors to τ_s and τ_{-s} are

$$V_s := \begin{pmatrix} \frac{1}{i\tau_s} v_s \\ v_s \end{pmatrix} \quad \text{and} \quad V_{-s} := \begin{pmatrix} \frac{1}{i\tau_{-s}} v_s \\ v_s \end{pmatrix}$$

resp. with $v_s(x) = \cos(\tau_s \rho x) + \frac{h_2}{\tau_s \rho} \cos(\tau_s \rho x)$.

- (2) The complex eigenvalues $i\mu$ and $\Leftrightarrow i\mu$ of A_2 are determined by the unique solution of

$$\tanh(\mu \rho N) = \Leftrightarrow \frac{\mu \rho}{h_2}, \quad \mu > 0. \quad (4.33)$$

As a rough estimation we have (cf. Fig. 4.2)

$$1 \geq \tanh(\mu \rho N) = \Leftrightarrow \frac{\mu \rho}{h_2} \Rightarrow \mu \leq \Leftrightarrow \frac{h_2}{\rho}$$

where the larger $|h_2|$ or smaller ρ are, the better is the estimation.

The eigenvectors for $i\mu$ and $\Leftrightarrow i\mu$ are resp.

$$\Psi_+ := \begin{pmatrix} \Leftrightarrow \frac{1}{\mu} e_\mu \\ e_\mu \end{pmatrix} \quad \text{and} \quad \Psi_- := \begin{pmatrix} \frac{1}{\mu} e_\mu \\ e_\mu \end{pmatrix}$$

with $e_\mu(x) = \cosh(\mu \rho x) + \frac{h_2}{\mu \rho} \sinh(\mu \rho x)$.

THE ENERGY FORM

The energy form

$$\left[\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \right]_2 := \frac{1}{2} \left(h_2 u_0 \bar{v}_0 + \int_0^N (u_0)_x \overline{(v_0)_x} + \rho^2 u_1 \bar{v}_1 dx \right) \quad (4.34)$$

is an indefinite metric: the vector $\begin{pmatrix} \frac{1}{\mu} e_\mu \\ 0 \end{pmatrix}$ has got the negative energy 'norm'

$$\left[\begin{pmatrix} \frac{1}{\mu} e_\mu \\ 0 \end{pmatrix}, \begin{pmatrix} \frac{1}{\mu} e_\mu \\ 0 \end{pmatrix} \right]_2 = \frac{1}{2} \frac{1}{\mu^2} \langle L_2 e_\mu, e_\mu \rangle_{\rho^2} = \Leftrightarrow \frac{1}{2} \langle e_\mu, e_\mu \rangle_{\rho^2} < 0$$

For some details on the theory of spaces with indefinite metrics see Appendix A.5.

It is checked easily that the orthogonal complement (w.r.t. the energy form $[\cdot, \cdot]_2$) of the one-dimensional subspace $\Sigma := \text{span}\left(\begin{smallmatrix} 1 \\ \mu \\ 0 \end{smallmatrix} e_\mu\right)$ is

$$\Sigma^{[\perp]} = \mathcal{C}_2[\ominus]\Sigma = \left\{ \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} : u_0 \perp_{\mathcal{L}_{2,\rho^2}} e_\mu \right\}.$$

L_2 is positive on $\Sigma^{[\perp]}$, so that $[\cdot, \cdot]_2$ is positive and defines a norm $\|\cdot\|_2$. Hence $(\mathcal{C}_2, [\cdot, \cdot]_2)$ is in fact a Pontrjagin space with an one-dimensional negative subspace Σ . The $[\cdot, \cdot]_2$ -orthogonal projection onto Σ is

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \mapsto \frac{1}{\|e_\mu\|_{\mathcal{L}_{2,\rho^2}}^2} \begin{pmatrix} \langle u_0, e_\mu \rangle_{\rho^2} e_\mu \\ 0 \end{pmatrix}$$

so that for the canonical symmetry J we get

$$J : \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \mapsto \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \Leftrightarrow 2 \frac{1}{\|e_\mu\|_{\mathcal{L}_{2,\rho^2}}^2} \begin{pmatrix} \langle u_0, e_\mu \rangle_{\rho^2} e_\mu \\ 0 \end{pmatrix}.$$

The positive inner product $[\cdot, \cdot]_J = [J\cdot, \cdot]_2$ defines a norm $\|\cdot\|_J$ on \mathcal{C}_2 , which we will use from now on.

THE NORMS OF THE EIGENVECTORS

The $\|\cdot\|_J$ -norms of $V_{\pm s}$, $s \in \mathbb{Z}^*$ are (see Appendix C.1.2)

$$\|V_s\|_J^2 = \|V_{-s}\|_J^2 = \|V_s\|_2^2 = \frac{\rho^2 N}{2} + h_2 \frac{N h_2 + 1}{2\tau_s^2} \geq \frac{\rho^2 N}{2}. \quad (4.35)$$

The above inequality is true, since

$$h_2 < \Leftrightarrow \frac{1}{N} \Rightarrow h_2(N h_2 + 1) > 0.$$

And for the norms of Ψ_+ and Ψ_- we get

$$\|\Psi_\pm\|_J^2 = \frac{\rho^2 N}{2} \Leftrightarrow h_2 \frac{N h_2 + 1}{2\mu^2}. \quad (4.36)$$

As one expects, the norms are positive since $\frac{1}{\mu^2} \geq \frac{\rho^2}{h_2^2}$ and then

$$\begin{aligned} \|\Psi_\pm\|_J^2 > 0 &\Leftrightarrow \frac{\rho^2 N}{2} > \frac{N h_2^2 + h_2}{2} \frac{1}{\mu^2} \\ &\Leftrightarrow \rho^2 N > (N h_2^2 + h_2) \frac{\rho^2}{h_2^2} \\ &\Leftrightarrow h_2 < 0 \quad (\text{true}). \end{aligned}$$

It also holds the estimation from above

$$\|\Psi_\pm\|_J^2 \leq \frac{\rho^2 N}{2} \Leftrightarrow \frac{N h_2^2 + h_2}{2} \frac{\rho^2}{h_2^2} = \Leftrightarrow \frac{\rho^2}{2h_2}. \quad (4.37)$$

THE RESOLVENT OF A_2

From (4.17) and (4.18) follows for $\alpha \in \mathbb{C}$ not an eigenvalue of A_2

$$\begin{aligned} l_2(x) &= \cos(\alpha\rho x) + \frac{h_2}{\alpha\rho} \sin(\alpha\rho x) \\ r(x) &= \cos(\alpha\rho x) \Leftrightarrow \cot(\alpha\rho N) \sin(\alpha\rho x). \end{aligned}$$

Then we have from (4.19) and (4.20)

$$\begin{aligned} G_2^\alpha(x, \xi) &= \begin{cases} \frac{1}{h_2 + \alpha\rho \cot(\alpha\rho N)} l_2(x) r(\xi) & , 0 \leq x \leq \xi \leq N \\ \frac{1}{h_2 + \alpha\rho \cot(\alpha\rho N)} l_2(\xi) r(x) & , 0 \leq \xi \leq x \leq N \end{cases} \\ K_2^\alpha &: y(\cdot) \mapsto \rho^2 \int_0^N G_2^\alpha(\cdot, \xi) y(\xi) d\xi, \end{aligned} \quad (4.38)$$

and from (4.8) the resolvent of A_2 defined on \mathcal{C}_2

$$R_2^\alpha \begin{pmatrix} f \\ g \end{pmatrix} = [A_2 \Leftrightarrow \alpha \text{id}_{\mathcal{C}_2}]^{-1} \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} \alpha K_2^\alpha & \Leftrightarrow K_2^\alpha \\ i + i\alpha^2 K_2^\alpha & \alpha K_2^\alpha \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix}. \quad (4.39)$$

THE DECOMPOSITION OF \mathcal{C}_2

Despite the fact that \mathcal{C}_2 is a Pontrjagin space, a more natural decomposition of \mathcal{C}_2 is into the two A_2 -invariant subspaces

$$\Pi_- := \text{span}(\Psi_+, \Psi_-) \quad \text{and} \quad \Pi_+ := \mathcal{C}_2[\ominus]\Pi_- = \text{span}\{V_s : s \in \mathbb{Z}^*\},$$

where Π_+ and Π_- are orthogonal both w.r.t $[\cdot, \cdot]_2$ and $[\cdot, \cdot]_J$ (this follows from (4.40) below). Define P_- the orthogonal projection onto Π_- and $P_+ := \text{id}_{\mathcal{C}_2} \Leftrightarrow P_-$. The P_\pm are orthogonal projections w.r.t. both forms $[\cdot, \cdot]_J$ and $[\cdot, \cdot]_2$. Also we have the property

$$\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \perp \Pi_- \Leftrightarrow u_0 \perp e_\mu \wedge u_1 \perp e_\mu.$$

Since $J = I$ in Π_+ it is $[\cdot, \cdot]_2 = [\cdot, \cdot]_J$ on Π_+ , whereas otherwise the following holds

$$[W, \Psi_+]_J = [W, \Psi_-]_2 \quad \wedge \quad [W, \Psi_-]_J = [W, \Psi_+]_2, \quad \forall W \in \mathcal{C}_2 \quad (4.40)$$

And because $V_t \perp \Pi_-$

$$[W, V_t]_J = [W, V_t]_2, \quad \forall W \in \mathcal{C}$$

Restricted onto $\Pi_+ \cap \mathcal{D}(A_2)$ the operator A_2 is self-adjoint for the same reason as A_1 is self-adjoint on $\mathcal{D}(A_1)$, and the set $\{V_s : s \in \mathbb{Z}^*\}$ is a complete and orthogonal set for Π_+ . The restriction $A_2|_{\Pi_-}$ is a two-dimensional operator which has the matrix form

$$A_2|_{\Pi_-} = \begin{pmatrix} 0 & \Leftrightarrow i \\ \Leftrightarrow i\mu^2 & 0 \end{pmatrix}.$$

w.r.t. the more convenient basis \mathcal{B} for Π_-

$$\mathcal{B} := \left\{ \begin{pmatrix} e_\mu \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e_\mu \end{pmatrix} \right\}.$$

$A_2|_{\Pi_-}$ is anti-hermitian w.r.t. $[\cdot, \cdot]_J$ (see Appendix C.1.3), that is

$$(A_2|_{\Pi_-})^* = \overline{(A_2|_{\Pi_-})^t} = \Leftrightarrow A_2|_{\Pi_-}.$$

Remark 4.3

The effect of changing the inner product is that A_2 loses its property of symmetricity on a two-dimensional subspace.

STABILITY

Now we show that the norm $\|\cdot\|_J$ is suitable to investigate the question of stability. If for a certain vector function $\mathcal{U}(t)$ the $\|\cdot\|_J$ -norm is uniformly bounded for all times $t \geq 0$ then the physical energy $S(t)$ of the string is bounded as well. To show this assume

$$\exists M > 0 : \|\mathcal{U}(t)\|_J^2 \leq M \quad \forall t \geq 0.$$

Decompose the Cauchy vector $\mathcal{U}(t)$ for given $t \geq 0$ into two mutual orthogonal components, that is

$$\mathcal{U}(t) = U_+(t) + U_-(t) = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} + \begin{pmatrix} a(t)e_\mu \\ b(t)e_\mu \end{pmatrix}, \quad a(t), b(t) \in \mathbb{C},$$

where $U_+(t) \in \Pi_+$ and $U_-(t) \in \Pi_-$, then certainly for all $t \geq 0$

$$\|U_+\|_J^2 = \|U_+\|_2^2 \leq M \quad \text{and} \quad \|U_-\|_J^2 \leq M.$$

(1) First we verify that

$$\exists M_1 > 0 : S_+(t) = \int_0^N |(u_0)_x|^2 + \rho^2 |u_1|^2 dx \leq M_1 \quad \forall t \geq 0.$$

We have

$$h_2 |u_0|^2|_0 + \int_0^N |(u_0)_x|^2 + \rho^2 |u_1|^2 dx \leq 2M \quad \forall t \geq 0, \quad (4.41)$$

but, as $h_2 < 0$, it could happen that both terms above in (4.41) are unbounded even though the inequality is fulfilled. However, this is not the case, as it is shown in the following.

$U_+(t)$ can be expanded w.r.t. the set $\{V_s : s \in \mathbb{Z}^*\}$ such that

$$U_+(t) = \sum_{s \in \mathbb{Z}^*} \alpha(t) V_s$$

with

$$\|U_+(t)\|_2^2 = \sum_{s \in \mathbb{Z}^*} |\alpha(t)|^2 \|V_s\|_2^2 < M < \infty.$$

From $\frac{\rho^2 N}{2} \leq \|V_s\|_2^2$ follows that

$$\sum_{s \in \mathbb{Z}^*} |\alpha(t)|^2 \leq \frac{2M}{\rho^2 N}.$$

Thus we get, since the sum $\sum_{s \in \mathbb{Z}^*} \frac{1}{|\tau_s|^2}$ converges ($|\tau_s| = o(1/|s|)$),

$$\begin{aligned} |h_2 \|u_0\|^2|_{x=0} &= |h_2| \left| \sum_{s \in \mathbb{Z}^*} \alpha_s(t) \frac{1}{i\tau_s} \right|^2 \\ &\leq |h_2| \sum_{s \in \mathbb{Z}^*} |\alpha_s(t)|^2 \sum_{s \in \mathbb{Z}^*} \frac{1}{|\tau_s|^2} \\ &\leq \frac{2M|h_2|}{\rho^2 N} \sum_{s \in \mathbb{Z}^*} \frac{1}{|\tau_s|^2} =: \hat{M}. \end{aligned}$$

Hence

$$S_+(t) = \int_0^N |(u_0)_x|^2 + \rho^2 |u_1|^2 dx \leq 2M + |h_2 \|u_0\|^2|_0 \leq 2M + \hat{M} =: M_1$$

is uniformly bounded for all $t \geq 0$.

(2) The same is true for the part in Π_- . We want to show that

$$\exists M_2 > 0 : S_-(t) = \int_0^N |a(t)|^2 |(e_\mu)_x|^2 + \rho^2 |b(t)|^2 |e_\mu|^2 dx \leq M_2 \quad \forall t \geq 0$$

It is

$$\|U_-(t)\|_J^2 = \frac{1}{2} (\mu^2 |a(t)|^2 + |b(t)|^2) \langle e_\mu, e_\mu \rangle_{\rho^2} \leq M \quad \forall t \geq 0,$$

so that especially

$$|a(t)|^2 \leq \frac{2M}{\mu^2} \langle e_\mu, e_\mu \rangle_{\rho^2} \quad \wedge \quad |b(t)|^2 \leq 2M \langle e_\mu, e_\mu \rangle_{\rho^2}$$

and therefore

$$S_-(t) \leq \langle e_\mu, e_\mu \rangle_{\rho^2} \int_0^N \frac{2M}{\mu^2} |(e_\mu)_x|^2 + \rho^2 2M |e_\mu|^2 dx =: M_2 \quad \forall t \geq 0.$$

Hence we get the desired result that

$$\begin{aligned} S(t) &= \int_0^N |(u_0(t) + a(t)e_\mu)_x|^2 + \rho^2 |u_1(t) + b(t)e_\mu|^2 dx \\ &\leq 2 \int_0^N |(u_0)_x|^2 + |a(t)|^2 |(e_\mu)_x|^2 + \rho^2 (|u_1|^2 + |b(t)|^2 |e_\mu|^2) dx \\ &\leq 2(S_+(t) + S_-(t)) \leq 2(M_1 + M_2) \quad \forall t \geq 0 \end{aligned}$$

is bounded uniformly.

4.2 Some relations between the generators of the stable and the unstable dynamics

4.2.1 The domains

The domains $\mathcal{D}(A_1)$ and $\mathcal{D}(A_2)$ are both dense (w.r.t. to the respective forms) subsets of the same space

$$\mathcal{C} := \mathcal{C}_1 = \mathcal{C}_2 = \left\{ \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} : u_0 \in W_2^1, u_1 \in \mathcal{L}_2, u_0|_N = 0 \right\}.$$

The intersection of $\mathcal{D}(A_1)$ and $\mathcal{D}(A_2)$ is the set

$$\begin{aligned} \mathcal{D}_0 &:= \mathcal{D}(A_1) \cap \mathcal{D}(A_2) \\ &= \left\{ \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} : u_0 \in W_2^2, u_1 \in W_2^1, (u_0)_x|_0 = u_0|_0 = 0, u_0|_N = 0, u_1|_N = 0 \right\}, \end{aligned}$$

and restricted onto \mathcal{D}_0 the operators A_1 and A_2 are equal

$$A_0 := A_1|_{\mathcal{D}_0} = A_2|_{\mathcal{D}_0}$$

and certainly symmetric.

In Appendix C.1.4 are given the calculations to verify that

$$\begin{aligned} U \in \mathcal{D}_0, V \in \mathcal{D}(A_1) &\Rightarrow [A_0 U, V]_1 = [U, A_1 V]_1 \\ \text{and } U \in \mathcal{D}_0, V \in \mathcal{D}(A_2) &\Rightarrow [A_0 U, V]_2 = [U, A_2 V]_2. \end{aligned}$$

So, when we denote with A_0^* the adjoint of A_0 w.r.t. the positive metric $[\cdot, \cdot]_1$ and with A_0^+ the adjoint w.r.t. the indefinite metric $[\cdot, \cdot]_2$, it follows

$$A_1 \subset A_0^* \quad \wedge \quad A_2 \subset A_0^+.$$

The difference between A_1 and A_2 is, basically, that both operators are self-adjoint extensions w.r.t. different inner products. The closure of the domains under the respective forms is the same space \mathcal{C} , considering the space \mathcal{C} as a set of vectors without any metric structure.

4.2.2 The energy forms

The energy forms $[\cdot, \cdot]_1$ and $[\cdot, \cdot]_2$ (also $[\cdot, \cdot]_J$) are defined on the same space \mathcal{C} , so it is possible to compare them. From (4.21) and (4.34) we immediately get for $\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \in \mathcal{C}$

$$\left[\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \right]_1 = \frac{1}{2} (h_1 \Leftrightarrow h_2) u_0|_0 \overline{(v_0|_0)} + \left[\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \right]_2,$$

and that the forms coincide on the intersection \mathcal{D}_0 of the domains. Also does follow $[U, U]_1 \geq [U, U]_2$. With regard to Lemma 2.26 it would be of particular interest to obtain equivalence, that is there exists $\gamma > 0$ such that $\|U\|_1 \leq \gamma \|U\|_J$. Since the norms $\|\cdot\|_1$ and $\|\cdot\|_J$ are equivalent on the finite-dimensional subspace Π_- , it is only required to show above inequality for all $U \in \Pi_+$.

4.2.3 The eigenvalues and eigenvectors

From (4.23) and (4.30) we get that for $s \in \mathbb{N}$

$$|\lambda_{s+1} \Leftrightarrow \tau_s| = |\lambda_{-s-1} \Leftrightarrow \tau_{-s}| = |\varepsilon_s + \delta_s|,$$

and in fact with (4.24) and (4.31)

$$|\lambda_{s+1} \Leftrightarrow \tau_s| = |\lambda_{-s-1} \Leftrightarrow \tau_{-s}| \leq \frac{2Nh_1}{(2s \Leftrightarrow 1)\pi} \Leftrightarrow \frac{Nh_2}{s\pi} \leq \frac{N}{s\pi}(h_1 \Leftrightarrow h_2). \quad (4.42)$$

Thus the differences of the eigenvalues approach to zero for $|s| \rightarrow \infty$, the same holds for the energy norms of the differences of the eigenvectors $U_{s+1} \Leftrightarrow V_s$ and $U_{-s-1} \Leftrightarrow V_{-s}$, that is

$$\begin{aligned} \lim_{s \rightarrow \infty} \|U_{s+1} \Leftrightarrow V_s\|_1^2 &= 0, \\ \lim_{s \rightarrow \infty} \|U_{-s-1} \Leftrightarrow V_{-s}\|_1^2 &= 0, \\ \lim_{s \rightarrow \infty} \|U_{s+1} \Leftrightarrow V_s\|_2^2 &= \lim_{s \rightarrow \infty} \left(\frac{h_2 - h_1}{\lambda_{s+1}\tau_s} + \|U_{s+1} \Leftrightarrow V_s\|_1^2 \right) = 0, \\ \lim_{s \rightarrow \infty} \|U_{-s-1} \Leftrightarrow V_{-s}\|_2^2 &= \lim_{s \rightarrow \infty} \left(\frac{h_2 - h_1}{\lambda_{-s-1}\tau_{-s}} + \|U_{-s-1} \Leftrightarrow V_{-s}\|_1^2 \right) = 0. \end{aligned} \quad (4.43)$$

For the calculations refer to Appendix C.1.5.

4.2.4 The eigenvector expansions

The sets of eigenvectors $\mathcal{U} := \{U_s : s \in \mathbb{Z}^*\}$ and $\mathcal{V} := \{V_s : s \in \mathbb{Z}^*\} \cup \{\Psi_+, \Psi_-\}$ both form complete orthogonal sets for the space \mathcal{C} (w.r.t. different inner products of course). Therefore we can expand each eigenvector $U \in \mathcal{U}$ w.r.t. \mathcal{V} and vice versa each $W \in \mathcal{V}$ w.r.t. \mathcal{U} , that is

$$U_s = \sum_{t \in \mathbb{Z}^*} c_s^t V_t + b_s^+ \Psi_+ + b_s^- \Psi_-, \quad s \in \mathbb{Z}^* \quad (4.44)$$

$$V_t = \sum_{s \in \mathbb{Z}^*} d_t^s U_s, \quad t \in \mathbb{Z}^*, \quad (4.45)$$

$$\Psi_+ = \sum_{s \in \mathbb{Z}^*} \psi_+^s U_s, \quad (4.46)$$

$$\Psi_- = \sum_{s \in \mathbb{Z}^*} \psi_-^s U_s, \quad (4.47)$$

Clearly we have for the coefficients

$$c_s^t = \frac{1}{\|V_t\|_J^2} [U_s, V_t]_J = \frac{1}{\|V_t\|_2^2} [U_s, V_t]_2, \quad (4.48)$$

$$b_s^+ = \frac{1}{\|\Psi_+\|_J^2} [U_s, \Psi_+]_J = \frac{1}{\|\Psi_+\|_2^2} [U_s, \Psi_-]_2, \quad (4.49)$$

$$b_s^- = \frac{1}{\|\Psi_+\|_J^2} [U_s, \Psi_-]_J = \frac{1}{\|\Psi_-\|_2^2} [U_s, \Psi_+]_2, \quad (4.50)$$

and

$$d_t^s = \frac{1}{\|U_s\|_1^2} [V_t, U_s]_1, \quad (4.51)$$

$$\psi_s^+ = \frac{1}{\|U_s\|_1^2} [\Psi_+, U_s]_1, \quad (4.52)$$

$$\psi_s^- = \frac{1}{\|U_s\|_1^2} [\Psi_-, U_s]_1. \quad (4.53)$$

In order to calculate the coefficients we need to know the products $[U_s, V_t]_{1/2}$ and $[U_s, \Psi_{\pm}]_J$, and for these we need to have some expressions for the products $\langle u_{|s|}, e_{\mu} \rangle_{\ell}$ and $\langle u_{|s|}, v_{|t|} \rangle_{\ell}$, since then for $s, t \in \mathbb{Z}^*$ we get

(a)

$$\begin{aligned} [\Psi_+, U_s]_2 &= \frac{1}{2} \left(\frac{1}{i\lambda_s\mu} \right) \langle \Leftrightarrow \mu^2 e_{\mu}, u_{|s|} \rangle_{\rho^2} + \langle e_{\mu}, u_{|s|} \rangle_{\rho^2} \\ &= \frac{1}{2} \left(1 + i \frac{\mu}{\lambda_s} \right) \langle e_{\mu}, u_{|s|} \rangle_{\rho^2}, \end{aligned} \quad (4.54)$$

and

$$\begin{aligned} [\Psi_+, U_s]_2 &= \frac{h_2 \Leftrightarrow h_1}{2i\mu\lambda_s} + \overline{[U_s, \Psi_+]_1} \\ &= \frac{h_2 \Leftrightarrow h_1}{2i\mu\lambda_s} + \frac{1}{2} \left(1 \Leftrightarrow i \frac{\lambda_s}{\mu} \right) \langle e_{\mu}, u_{|s|} \rangle_{\rho^2}. \end{aligned} \quad (4.55)$$

We subtract both equations and get

$$\langle e_{\mu}, u_{|s|} \rangle_{\rho^2} = \frac{h_1 \Leftrightarrow h_2}{\lambda_s^2 + \mu^2}. \quad (4.56)$$

(b) Similarly we have

$$[V_t, U_s]_2 = \frac{1}{2} \left(1 + \frac{\tau_t}{\lambda_s} \right) \langle v_{|t|}, u_{|s|} \rangle_{\rho^2}, \quad (4.57)$$

and

$$\begin{aligned} [V_t, U_s]_2 &= \frac{h_2 \Leftrightarrow h_1}{2\lambda_s\tau_t} + [U_s, V_t]_1 \\ &= \frac{h_2 \Leftrightarrow h_1}{2\lambda_s\tau_t} + \frac{1}{2} \left(1 + \frac{\lambda_s}{\tau_t} \right) \langle v_{|t|}, u_{|s|} \rangle_{\rho^2}. \end{aligned} \quad (4.58)$$

Again subtracting both equations we get

$$\langle v_{|t|}, u_{|s|} \rangle_{\rho^2} = \frac{h_1 \Leftrightarrow h_2}{(\lambda_s \Leftrightarrow \tau_t)(\lambda_s + \tau_t)}. \quad (4.59)$$

Summarising (4.54)-(4.59) we get

$$[U_s, \Psi_+]_J = \frac{1}{2} \left(1 + i \frac{\mu}{\lambda_s} \right) \langle e_\mu, u_s \rangle_{\rho^2} = \frac{h_1 \Leftrightarrow h_2}{2\lambda_s(\lambda_s \Leftrightarrow i\mu)} \quad (4.60)$$

$$[U_s, \Psi_-]_J = \frac{1}{2} \left(1 \Leftrightarrow i \frac{\mu}{\lambda_s} \right) \langle e_\mu, u_s \rangle_{\rho^2} = \frac{h_1 \Leftrightarrow h_2}{2\lambda_s(\lambda_s + i\mu)} \quad (4.61)$$

$$[\Psi_+, U_s]_1 = \text{conj} \left(\frac{1}{2} \left(1 + i \frac{\lambda_s}{\mu} \right) \langle e_\mu, u_s \rangle_{\rho^2} \right) = \Leftrightarrow i \frac{h_1 \Leftrightarrow h_2}{2\mu(\lambda_s \Leftrightarrow i\mu)} \quad (4.62)$$

$$[\Psi_-, U_s]_1 = \text{conj} \left(\frac{1}{2} \left(1 \Leftrightarrow i \frac{\lambda_s}{\mu} \right) \langle e_\mu, u_s \rangle_{\rho^2} \right) = i \frac{h_1 \Leftrightarrow h_2}{2\mu(\lambda_s + i\mu)} \quad (4.63)$$

$$[V_t, U_s]_1 = \begin{cases} \frac{1}{2} \left(1 + \frac{\lambda_s}{\tau_t} \right) \langle u_s, v_t \rangle_{\rho^2} & = \frac{h_1 - h_2}{2\tau_t(\lambda_s - \tau_t)} \\ \frac{h_1 - h_2}{2\lambda_s \tau_t} + \frac{1}{2} \left(1 + \frac{\tau_t}{\lambda_s} \right) \langle u_s, v_t \rangle_{\rho^2} & = \frac{h_1 - h_2}{2\lambda_s \tau_t} + \frac{h_1 - h_2}{2\lambda_s(\lambda_s - \tau_t)} \end{cases} \quad (4.64)$$

$$[U_s, V_t]_2 = \begin{cases} \frac{1}{2} \left(1 + \frac{\tau_t}{\lambda_s} \right) \langle u_s, v_t \rangle_{\rho^2} & = \frac{h_1 - h_2}{2\lambda_s(\lambda_s - \tau_t)} \\ \frac{h_2 - h_1}{2\lambda_s \tau_t} + \frac{1}{2} \left(1 + \frac{\lambda_s}{\tau_t} \right) \langle u_s, v_t \rangle_{\rho^2} & = \Leftrightarrow \frac{h_1 - h_2}{2\lambda_s \tau_t} + \frac{h_1 - h_2}{2\tau_t(\lambda_s - \tau_t)} \end{cases} \quad (4.65)$$

Now we need only substitute (4.60)-(4.65) into (4.48)-(4.53) in order to obtain the expressions for the coefficients. One will see that, for $s, t \in \mathbb{N}$

$$\begin{aligned} b_{-s}^+ &= b_s^- = \text{conj}(b_{-s}^-) = \text{conj}(b_s^+) \\ \psi_+^{-s} &= \psi_-^s = \text{conj}(\psi_+^s) = \text{conj}(\psi_-^{-s}) \end{aligned} \quad (4.66)$$

and

$$\begin{aligned} c_{-s}^{-t} &= c_s^t, & c_s^{-t} &= c_{-s}^t \\ d_{-t}^{-s} &= d_t^s, & d_t^{-s} &= d_{-t}^s \end{aligned} \quad (4.67)$$

Hence it suffices to calculate b_s^+ and ψ_+^s only for positive s , c_s^t and d_t^s only for $s > 0 \wedge t > 0$ as well as for $s > 0 \wedge t < 0$. We also make the following observations:

For $s > 0$ we have with the Cauchy-Schwarz inequality ($[\cdot, \cdot]_2$ is positive on Π_+)

$$\begin{aligned} |c_{s+1}^s \Leftrightarrow 1| &= \left| \frac{[U_{s+1} - V_s, V_s]_2}{\|V_s\|_2^2} \right| \leq \sqrt{\frac{1}{\rho^2 N}} \|U_{s+1} \Leftrightarrow V_s\|_2, \\ |d_s^{s+1} \Leftrightarrow 1| &= \left| \frac{[V_s - U_{s+1}, U_{s+1}]_1}{\|U_{s+1}\|_1^2} \right| \leq \sqrt{\frac{1}{\rho^2 N}} \|U_{s+1} \Leftrightarrow V_s\|_1. \end{aligned}$$

Thus with (4.43)

$$\begin{aligned} \lim_{s \rightarrow \infty} c_{s+1}^s &= 1 = \lim_{s \rightarrow \infty} c_{-s-1}^{-s} \\ \text{and } \lim_{s \rightarrow \infty} d_s^{s+1} &= 1 = \lim_{s \rightarrow \infty} d_{-s}^{-s-1}. \end{aligned} \quad (4.68)$$

But for all $l \in \mathbb{Z} \setminus \{1\}$ we have

$$\lim_{s \rightarrow \pm\infty} c_{s \pm l}^s = 0 \quad \text{and} \quad \lim_{s \rightarrow \pm\infty} d_s^{s \pm l} = 0. \quad (4.69)$$

Remark 4.4

(1) It is not surprising that $\lim_{|t| \rightarrow \infty} c_s^t = 0$ and $\lim_{|s| \rightarrow \infty} d_t^s = 0$, which follow from

the Riemann lemma for Fourier coefficients. The significance of the other limits obtained in (4.68) and (4.69) is that the spaces $\text{span}\{U_s : |s| \geq l+1\}$ are almost orthogonal to the spaces $\text{span}\{V_s : |s| \leq l \Leftrightarrow 1\}$ for large $l \in \mathbb{N}$. For large $s > 0$ the eigenvalues λ_{s+1} and τ_s are also almost the same, respectively for large $s < 0$ the eigenvalues λ_{s-1} and τ_s . Thus one can say that the operators A_1 and A_2 basically differ only on the complements of the spaces $\text{span}\{U_s : |s| \geq l+1\} \cong \text{span}\{V_s : |s| \geq l\}$ for large l . This complement is of finite dimension.

- (2) Above property is important for the motivation to apply the Galerkin method in Section 4.4 to approximate the spectrum.
- (3) The calculations of the coefficients of the eigenvector expansions do not take into account the special values of the tension T and the density function $\varrho(x)$ of the solvable model, thus they are valid also for the general case.

4.2.5 The difference of the resolvents

Both resolvents R_1^α and R_2^α (if they exist for that α) are defined on the same space \mathcal{C} , so it is possible to consider the difference $R_2^\alpha \Leftrightarrow R_1^\alpha$. We have with (4.28) and (4.39)

$$R_2^\alpha \Leftrightarrow R_1^\alpha = \begin{pmatrix} \alpha & \Leftrightarrow i \\ i\alpha^2 & \alpha \end{pmatrix} (K_2^\alpha \Leftrightarrow K_1^\alpha),$$

and for the difference of the Green's functions (see (4.27) and (4.38)) we get for $0 \leq x \leq \xi \leq N$ with Lemma A.43

$$\begin{aligned} G_2^\alpha(x, \xi) \Leftrightarrow G_1^\alpha(x, \xi) &= \frac{l_2(x)r(\xi)}{W(l_2, r)} \Leftrightarrow \frac{l_1(x)r(\xi)}{W(l_1, r)} \\ &= r(\xi) \frac{l_2(x) [l_1(x)r'(x) \Leftrightarrow l_1'(x)r(x)] \Leftrightarrow l_1(x) [l_2(x)r'(x) \Leftrightarrow l_2'(x)r(x)]}{W(l_2, r)W(l_1, r)} \\ &= \frac{W(l_1, l_2)}{W(l_2, r)W(l_1, r)} r(x)r(\xi). \end{aligned}$$

The same holds for $0 \leq \xi \leq x \leq N$, so that $K_2^\alpha \Leftrightarrow K_1^\alpha$ is the symmetric one-dimensional operator

$$\hat{K} := K_2^\alpha \Leftrightarrow K_1^\alpha : y(\cdot) \mapsto \Omega \int_0^N \varrho(\xi)r(\xi)y(\xi)d\xi \quad r(\cdot)$$

with

$$\Omega := \frac{W(l_1, l_2)}{W(l_2, r)W(l_1, r)}.$$

The action of $\hat{R}^\alpha := R_2^\alpha \Leftrightarrow R_1^\alpha$ is expressed by

$$\begin{aligned} \hat{R}^\alpha \begin{pmatrix} f \\ g \end{pmatrix} &= \Omega \begin{pmatrix} \alpha & \Leftrightarrow i \\ i\alpha^2 & \alpha \end{pmatrix} \begin{pmatrix} \langle f, r \rangle_\varrho \\ \langle g, r \rangle_\varrho \end{pmatrix}_r \\ \Rightarrow \hat{R}^\alpha \begin{pmatrix} f \\ g \end{pmatrix} &= \Omega \langle \alpha f \Leftrightarrow ig, r \rangle_\varrho \begin{pmatrix} r \\ i\alpha r \end{pmatrix}, \end{aligned} \tag{4.70}$$

which is also a one-dimensional operator mapping onto the subspace

$$\Lambda^\alpha := \text{span} \begin{pmatrix} r \\ i\alpha r \end{pmatrix}.$$

Since

$$\hat{R}^\alpha \begin{pmatrix} r \\ i\alpha r \end{pmatrix} = \Omega \langle 2\alpha r, r \rangle_\ell \begin{pmatrix} r \\ i\alpha r \end{pmatrix}$$

an estimate of the norm of \hat{R}^α is

$$\|\hat{R}^\alpha\|_{op} \geq 2|\alpha\Omega| \|r\|_{\mathcal{L}_{2,\ell}}^2.$$

The constants in (4.70) for the solvable model are

$$\Omega = \frac{h_1 \Leftrightarrow h_2}{[h_1 + \alpha\rho \cot(\alpha\rho N)][h_2 + \alpha\rho \cot(\alpha\rho N)]}$$

and $r(x) = \cos(\alpha\rho x) \Leftrightarrow \cot(\alpha\rho N) \sin(\alpha\rho x).$

4.3 The generator \mathcal{A} of the expectation semi-group

Now we consider the Cauchy problem (4.9) for which the value of h in the boundary condition (4.11) at $x = 0$ jumps between the two values $h_1 > 0$ and $h_2 < \Leftrightarrow \frac{1}{N}$. Let the jump process be realised by a symmetric, 2-state, continuous-time Markov process with exponentially distributed (intensity \varkappa) sojourn times (cf. Chapter 2) and generator

$$Q := \varkappa \begin{pmatrix} \Leftrightarrow 1 & 1 \\ 1 & \Leftrightarrow 1 \end{pmatrix}, \quad \varkappa > 0.$$

Then on the space $(\mathcal{C}, [.,.]_1) \times (\mathcal{C}, [.,.]_J)$ the infinitesimal generator $\mathcal{A}(\varkappa)$ of the expectation semi-group is

$$\mathcal{A}(\varkappa) = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} + i\varkappa \begin{pmatrix} \text{id}_{\mathcal{C}} & \Leftrightarrow \text{id}_{\mathcal{C}} \\ \Leftrightarrow \text{id}_{\mathcal{C}} & \text{id}_{\mathcal{C}} \end{pmatrix}$$

Alternatively we can make use of the decomposition of $\mathcal{C}_2 = \Pi_+ \oplus \Pi_-$ (cf. Section 4.1.4.) and consider the expectation semi-group on the space

$$(\mathcal{C}, [.,.]_1) \times [(\Pi_+, [.,.]_2) \times (\Pi_-, [.,.]_J)].$$

Define the imbeddings

$$\hat{P}_+ : \Pi_+ \hookrightarrow \mathcal{C} \quad \text{and} \quad \hat{P}_- : \Pi_- \hookrightarrow \mathcal{C},$$

the identities id_+ on Π_+ and id_- on Π_- , and recall that P_+ and P_- are the $[.,.]_J$ -orthogonal projections of \mathcal{C} onto Π_+ and Π_- respectively. Then $\mathcal{A}(\varkappa)$ is

$$\mathcal{A}(\varkappa) = \begin{pmatrix} A_1 + i\varkappa \text{id}_{\mathcal{C}} & \Leftrightarrow i\varkappa \hat{P}_+ & \Leftrightarrow i\varkappa \hat{P}_- \\ \Leftrightarrow i\varkappa P_+ & A_2|_{\Pi_+} + i\varkappa \text{id}_+ & 0 \\ \Leftrightarrow i\varkappa P_- & 0 & A_2|_{\Pi_-} + i\varkappa \text{id}_- \end{pmatrix}$$

The question to be pursued here is, again, what sufficient and/or necessary conditions there are so that for some values of \varkappa stability in average is obtained. And if so, we want to know, what these values are. Thus we need to show that under certain conditions the generator $\mathcal{A}(\varkappa)$ is dissipative or similar to a dissipative operator (cf. Theorem A.26 and Proposition A.29), or its spectrum is contained in the open UHP.

With regard to Hypothesis 3.1 we make the

Hypothesis 4.5

A necessary condition on the possible choice of values for \varkappa to obtain stability in average includes a relation between the two terms $|h_1 + \frac{1}{N}|$ and $|h_2 + \frac{1}{N}|$. Possibly the requirement is

$$\frac{h_1 + 1/N}{h_2 + 1/N} > 1$$

Further, we make the following assumption, which will be the basis for most of the approaches to determine the spectrum of \mathcal{A} . It is connected with the possibility to approximate the spectrum.

Assumption 4.6

For the operator \mathcal{A} the expression

$$\Delta = \frac{\|(\mathcal{A} \Leftrightarrow \alpha \text{id}_{\mathcal{C}})u\|}{\|u\|}$$

with $\alpha \in \mathbb{C}$ and $u \in H \times H$ is a good indication of the distance between α and the spectrum $\sigma(\mathcal{A})$. In particular, if Δ is small, then the distance $\text{dist}(\alpha, \sigma(\mathcal{A}))$ is small as well.

\mathcal{A} is not self-adjoint (cf. Remark 2.35(1)), so Corollary A.15 is not applicable. However, Corollary A.18 gives the relation between Δ and the distance to the numerical range. Also does the term Δ occur in the definition of the approximate spectrum (see Lemma A.6). On these grounds stands the assumption.

4.3.1 Asymptotic behaviour of the spectrum of \mathcal{A}

It is possible to describe the asymptotic behaviour of the approximate eigenvalues of $\mathcal{A}(\varkappa)$ for large absolute values. Recall that $U_{\pm(s+1)}$ ($s \in \mathbb{N}$) are the eigenvectors of A_1 for the eigenvalues $\lambda_{\pm(s+1)}$ with $\lambda_{\pm(s+1)} \approx \pm \frac{2s+1}{2}\pi$ for large s , and $V_{\pm s}$ are the eigenvectors of A_2 for the eigenvalues $\tau_{\pm s} \approx \pm \frac{2s+1}{2}\pi$. In Section 4.2 it is shown that

$$\lim_{s \rightarrow \infty} |\lambda_{\pm(s+1)} \Leftrightarrow \tau_{\pm s}| = 0, \quad (4.71)$$

and also that (with the calculations given in Appendix C.1.5)

$$\lim_{s \rightarrow \infty} \|U_{\pm(s+1)} \Leftrightarrow V_{\pm s}\|_j = 0, \quad j = 1, 2 \quad (4.72)$$

The complex numbers λ_{s+1} , $\lambda_{s+1} + 2i\varkappa$, λ_{-s-1} and $\lambda_{-s-1} + 2i\varkappa$ are approximate eigenvalues with approximate eigenvectors

$$\Phi_1(s) = \begin{pmatrix} U_{s+1} \\ V_s \end{pmatrix}, \quad \Phi_2(s) = \begin{pmatrix} U_{s+1} \\ -V_s \end{pmatrix}, \quad \Phi_3(s) = \begin{pmatrix} U_{-s-1} \\ V_{-s} \end{pmatrix}, \quad \text{and} \quad \Phi_4(s) = \begin{pmatrix} U_{-s-1} \\ -V_{-s} \end{pmatrix}.$$

as we will see in the following. The norms of above vectors satisfy

$$\|\Phi_j(s)\|^2 = \|U_{s+1}\|_1^2 + \|V_s\|_2^2 \geq \rho^2 N \quad j = 1, 2, 3, 4,$$

and it is

$$\mathcal{A}\Phi_1(s) \Leftrightarrow \lambda_{s+1}\Phi_1(s) = \begin{pmatrix} i\varkappa(U_{s+1} \Leftrightarrow V_s) \\ \Leftrightarrow i\varkappa(U_{s+1} \Leftrightarrow V_s) + (\tau_s \Leftrightarrow \lambda_{s+1})V_s \end{pmatrix}.$$

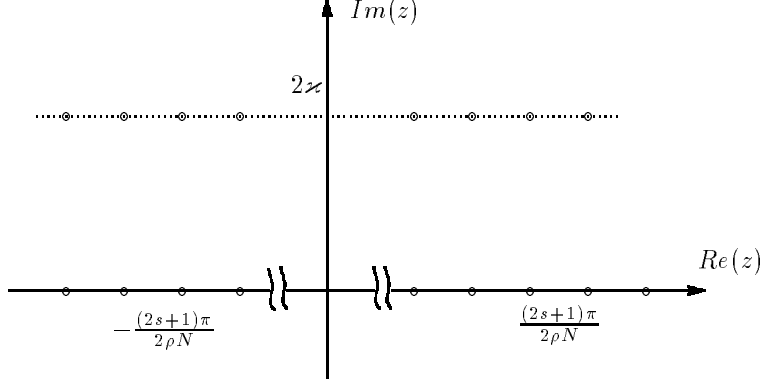


Figure 4.5: An asymptotic set for the spectrum $\sigma(\mathcal{A})$.

Therefore

$$\begin{aligned} \frac{\|\mathcal{A}\Phi_1(s) \Leftrightarrow \lambda_{s+1}\Phi_1(s)\|^2}{\|\Phi_1(s)\|^2} &\leq \\ &\leq \frac{1}{\rho^2 N} \left[\kappa^2 \|U_{s+1} \Leftrightarrow V_s\|_1^2 + (\kappa \|U_{s+1} \Leftrightarrow V_s\|_2^2 + |\tau_s \Leftrightarrow \lambda_{s+1}| \|V_s\|_2)^2 \right] \\ &\leq \frac{1}{\rho^2 N} \left[\kappa^2 \|U_{s+1} \Leftrightarrow V_s\|_1^2 + 2\kappa^2 \|U_{s+1} \Leftrightarrow V_s\|_2^2 + 2|\tau_s \Leftrightarrow \lambda_{s+1}|^2 \|V_s\|_2^2 \right]. \end{aligned}$$

With (4.71) and (4.72) we get

$$\lim_{s \rightarrow \infty} \frac{\|\mathcal{A}\Phi_1(s) \Leftrightarrow \lambda_{s+1}\Phi_1(s)\|}{\|\Phi_1(s)\|} = 0,$$

and the same limits for the expression with the vectors $\Phi_2(s)$, $\Phi_3(s)$ and $\Phi_4(s)$.

Above results show that one can expect that the set

$$\Lambda = \left\{ z : \operatorname{Re}(z) = \pm \frac{1}{\rho N} \frac{2s+1}{2} \pi \wedge \left(\operatorname{Im}(z) = 0 \vee \operatorname{Im}(z) = 2i\kappa \right) \right\} \quad (4.73)$$

is an asymptotic set for the approximate spectrum of $\mathcal{A}(\kappa)$ for large absolute values, illustrated in Fig. 4.5, accumulation points are $\Leftrightarrow \infty$ and $+\infty$.

Of essential importance for the question of stability is whether the imaginary parts of the eigenvalues near the real axis are positive. We know from the Section 2.4.2 that initially, for small κ , the imaginary parts of all eigenvalues near the real values $\lambda_{\pm s}$ and $\tau_{\pm s}$ are positive. However, nothing is known about the eigenvalues with small absolute value and for large κ .

4.3.2 Perturbation Theory and Stability in Average

It is not shown yet, that $Q^t \times \operatorname{id}_H$ is bounded. However, if it is bounded, the norm is bounded from below by $\|Q^t \times \operatorname{id}_H\| \geq \kappa$, see Remark 2.35(3). From the result of the study of the simple example in Chapter 3 we expect that one has to choose κ larger than some constant, say $\kappa \geq c$, in order to have stability. In any case we have then

$$\|Q^t \times \text{id}_H\| \geq c.$$

The theorems of Perturbation Theory like Proposition A.32 and Theorem A.36 estimate the distance between the spectra of the perturbed operator and of the unperturbed operator. The spectrum of $\mathcal{A}(0)$ is real with two complex eigenvalues. Knowing that the spectrum is slightly perturbed from $\sigma(\mathcal{A}(0))$, Proposition A.32 does not help to answer the question of stability. For that one has to show that the new eigenvalues, or non-regular points, are not only inside a small disc around the eigenvalues of $\mathcal{A}(0)$, but inside the *upper* half disc, in the UHP. That the eigenvalues for small values of \varkappa actually do 'move up' (cf. Section 2.4.2) is again not of much help - we expect $\varkappa \geq c > 0$ bounded away from zero. More detailed calculation on the series expansions are needed, maybe with the help of the *Feynman diagrammes*, (see R.D.Mattuk: "A Guide to Feynman Diagrams in the Many-Body Problem", McGraw-Hill Publishing Company Ltd, 1967).

Theorem A.36 is not applicable either. Here is given only an upper bound for the norms of the operators of the expectation semi-group, for bounded $Q^t \times \text{id}_H$. Since the exponential rate in the estimation of the norm $\|e^{i\mathcal{A}(0)t}\| \leq Ke^{\mu t}$ of the expectation semi-group generated by $\mathcal{A}(0)$ is already $\mu > 0$ from the accretive part of A_2 , we need to find a way to *decrease* it to zero, rather than estimate the increase.

Much stronger results of Perturbation Theory than the standard ones are required to find conditions for stability.

4.3.3 The resolvent of the generator \mathcal{A}

Recall from the calculation of the resolvent of \mathcal{A} in Section 2.4.4, that if α is in the spectrum of \mathcal{A} then one of the operators $R_1^{\alpha-i\varkappa}$, $R_2^{\alpha-i\varkappa}$, $T_1^{-1} := [A_2 \Leftrightarrow (\alpha \Leftrightarrow i\varkappa) \text{id}_C + \varkappa^2 R_1^{\alpha-i\varkappa}]^{-1}$, or $T_2^{-1} := [A_1 \Leftrightarrow (\alpha \Leftrightarrow i\varkappa) \text{id}_C + \varkappa^2 R_2^{\alpha-i\varkappa}]^{-1}$ does not exist.

- Assume, $R_1^{\alpha-i\varkappa}$ does not exist. Then $\alpha \Leftrightarrow i\varkappa$ is an eigenvalue of A_1 . Since all eigenvalues λ_s of A_1 are real it follows that $\alpha = \lambda_s + i\varkappa$ has imaginary part $\varkappa > 0$, i.e. lies in UHP, as required for stability.
- Assume, $R_2^{\alpha-i\varkappa}$ does not exist, thus $\alpha \Leftrightarrow i\varkappa$ is an eigenvalue of A_2 . That means now that either $\alpha = \tau_s + i\varkappa$ in the UHP or $\alpha = \pm i\mu + i\varkappa$. If we choose $\varkappa > \mu$, then all eigenvalues lie in the UHP.
- Now assume that the spectrum of T_1 includes the number zero. T_1 can be regarded as a perturbation of $T_1^0 = A_2 \Leftrightarrow (\alpha \Leftrightarrow i\varkappa) \text{id}_C + \varkappa^2 R_2^{\alpha-i\varkappa}$ by $\varkappa^2 \hat{R}^{\alpha-i\varkappa}$ since

$$A_2 \Leftrightarrow (\alpha \Leftrightarrow i\varkappa) \text{id}_C + \varkappa^2 R_1^{\alpha-i\varkappa} = A_2 \Leftrightarrow (\alpha \Leftrightarrow i\varkappa) \text{id}_C + \varkappa^2 R_2^{\alpha-i\varkappa} \Leftrightarrow \varkappa^2 \hat{R}^{\alpha-i\varkappa}.$$

And if the norm of the operator $\varkappa^2 \hat{R}^{\alpha-i\varkappa} = \varkappa^2 (R_2^{\alpha-i\varkappa} \Leftrightarrow R_1^{\alpha-i\varkappa})$ is small the distance of the spectrum of T_1 to the one of T_1^0 is small.

The eigenvalues α_s , $s \in \mathbb{Z}^*$ of T_0 satisfy the equation

$$\begin{aligned} 0 &= \lambda_s \Leftrightarrow (\alpha_s \Leftrightarrow i\varkappa) + \varkappa^2 \frac{1}{\lambda_s \Leftrightarrow (\alpha_s \Leftrightarrow i\varkappa)} \\ \Leftrightarrow 0 &= [\alpha + (\lambda_s + i\varkappa)]^2 + \varkappa^2 \\ \Leftrightarrow 0 &= \alpha^2 \Leftrightarrow 2(\lambda_s + i\varkappa)\alpha + (\lambda_s + i\varkappa)^2 + \varkappa^2 \end{aligned} \tag{4.74}$$

The discriminant of above quadratic equation is

$$4(\lambda_s + i\kappa)^2 \Leftrightarrow 4(\lambda_s^2 + 2i\kappa\lambda_s) = \Leftrightarrow 4\kappa^2,$$

so that the solutions of (4.74) are

$$\alpha_s^{(1)} = \lambda_s + 2i\kappa \quad \text{and} \quad \alpha_s^{(2)} = \lambda_s$$

with non-negative imaginary parts.

- The analogous calculations hold for the case that T_2^{-1} does not exist. However, now we get, additional to the zeroes τ_s and $\tau_s + 2i\kappa$, the zeroes $\pm i\mu$ and $\pm i\mu + 2i\kappa$.

Remark 4.7

- (1) The assumption we made above is that the norm of $\kappa^2 \hat{R}^{\alpha - i\kappa}$ is small.
- (2) Even if one can show that the assumption is correct, one still has to show that the perturbed eigenvalues of T_1 (or T_2) near the real values $\alpha_s^{(2)}$ are in the UHP.
- (3) Observe that above calculations give the same approximation of the spectrum as obtained from the investigation of the asymptotic behaviour.

4.4 Numerical approximation of the spectrum of \mathcal{A}

4.4.1 Preliminaries about the Galerkin method

As described in Mikhlin[23] the Bubnov-Galerkin method is a procedure to approximate eigenvalues of an operator A . It is a generalisation of the Ritz method, and it is applicable to operators A in Hilbert space which are not semi-bounded.

In order to solve the eigenvalue problem

$$A\Phi = \alpha\Phi, \quad \alpha \in \mathbb{C}$$

we choose a complete (not necessarily orthogonal) sequence of vectors $(\varphi_k)_{k=1}^{\infty}$ and set for $K \in \mathbb{N}$

$$\Phi_K := \sum_{k=1}^K a_k \varphi_k.$$

To determine the coefficients a_k we require that $A\Phi_K \Leftrightarrow \alpha\Phi_K$ is orthogonal to the vectors $\varphi_1, \dots, \varphi_K$, that is

$$\langle A\Phi_K \Leftrightarrow \alpha\Phi_K, \varphi_j \rangle = 0 \quad \forall j = 1, \dots, K \tag{Gal}$$

$$\Leftrightarrow \sum_{k=1}^{\infty} a_k \langle A\varphi_k \Leftrightarrow \alpha\varphi_k, \varphi_j \rangle = 0 \quad \forall j = 1, \dots, K \tag{4.75}$$

We get a system of K linear homogeneous equations in the variables a_1, \dots, a_K . The values for α for which there exists a non-trivial vector (a_1, \dots, a_K) can serve as approximations for the eigenvalues of A . These values are determined by setting the

determinant of the system (4.75) to zero, i.e.

$$\begin{vmatrix} \langle A\varphi_1 \Leftrightarrow \alpha\varphi_1, \varphi_1 \rangle & \cdots & \langle A\varphi_K \Leftrightarrow \alpha\varphi_K, \varphi_1 \rangle \\ \vdots & \ddots & \vdots \\ \langle A\varphi_1 \Leftrightarrow \alpha\varphi_1, \varphi_K \rangle & \cdots & \langle A\varphi_K \Leftrightarrow \alpha\varphi_K, \varphi_K \rangle \end{vmatrix} = 0.$$

The result is a polynomial of degree K , so that we obtain K solutions $\alpha_K^{(1)}, \dots, \alpha_K^{(K)}$ and corresponding eigenvectors $\Phi_1^{(1)}, \dots, \Phi_K^{(K)}$. In order to verify, that the numbers $\alpha_K^{(j)}$ actually approximate the eigenvalues of A one can apply Lemma A.6. If we can show that

(i) the value of

$$\Delta_K^{(j)} := \frac{\|A\Phi_K^{(j)} \Leftrightarrow \alpha_K^{(j)}\Phi_K^{(j)}\|}{\|\Phi_K^{(j)}\|} \quad (4.76)$$

approaches to zero, and that

(ii) the sequence of sets $(\Lambda_K)_{K \in \mathbb{N}} = (\{\alpha_K^{(j)}, j = 1, \dots, K\})_{K \in \mathbb{N}}$ contains convergent sequences $(\alpha_k)_{k \in \mathbb{N}}$ with $\alpha_k \in \chi_k$,

then the sets Λ_K approximate (at least finite parts of) the approximate spectrum.

4.4.2 The Galerkin method applied to the generator \mathcal{A}

For the generator

$$\mathcal{A} = \begin{pmatrix} A_1 + i\kappa \text{id}_{\mathcal{C}} & \Leftrightarrow i\kappa \text{id}_{\mathcal{C}} \\ \Leftrightarrow i\kappa \text{id}_{\mathcal{C}} & A_2 + i\kappa \text{id}_{\mathcal{C}} \end{pmatrix}$$

of the expectation semi-group there exists a canonical way of choosing the infinite complete set $\{\varphi_k\}_{k \in \mathbb{N}}$. We make use of the fact that the eigenvectors of A_1 and A_2 are known and form orthogonal complete sets and define

$$\{\varphi_k : k \in \mathbb{N}\} = \left\{ \begin{pmatrix} U_s \\ 0 \end{pmatrix}, s \in \mathbb{Z}^* \right\} \cup \left\{ \begin{pmatrix} 0 \\ V_s \end{pmatrix}, s \in \mathbb{Z}^* \right\} \cup \left\{ \begin{pmatrix} 0 & 0 \\ \Psi_+ & \Psi_- \end{pmatrix} \right\}.$$

Then we set as the vectors $\Phi^{(K)}$

$$\Phi_K = \sum_{s=-K-1, s \neq 0}^{K+1} E^s \begin{pmatrix} U_s \\ 0 \end{pmatrix} + \sum_{t=-K, t \neq 0}^K F^t \begin{pmatrix} 0 \\ V_t \end{pmatrix} + G^+ \begin{pmatrix} 0 \\ \Psi_+ \end{pmatrix} + G^- \begin{pmatrix} 0 \\ \Psi_- \end{pmatrix}. \quad (4.77)$$

Remark 4.8

We observed in Remark 4.4 that the two invariant spaces $\text{span}\{U_s, |s| \geq K+2\}$ and $\text{span}\{V_s, |s| \geq K+1\}$ w.r.t. the evolutions $e^{iA_1 t}$ and $e^{iA_2 t}$ respectively are 'almost' the same for large K . The mixing of the two evolutions, which is the effect of the Markov chain, basically affects only the finite dimensional complements $\text{span}\{U_s, |s| \leq K+1\}$ and $\text{span}\{V_s, |s| \geq K\} \cup \{\Psi_+, \Psi_-\}$. The procedure described in this section is an attempt to make use of this observation, the choice of the vector Φ_K is thus motivated.

For the Φ_K the condition (Gal) transforms into

$$\begin{aligned}
\left\langle \mathcal{A}\Phi_K \Leftrightarrow \alpha\Phi_K, \begin{pmatrix} U_s \\ 0 \end{pmatrix} \right\rangle &= 0, \quad s = \Leftrightarrow K \Leftrightarrow 1, \dots, K+1, s \neq 0 \\
\left\langle \mathcal{A}\Phi_K \Leftrightarrow \alpha\Phi_K, \begin{pmatrix} 0 \\ V_t \end{pmatrix} \right\rangle &= 0, \quad t = \Leftrightarrow K, \dots, K, t \neq 0 \\
\left\langle \mathcal{A}\Phi_K \Leftrightarrow \alpha\Phi_K, \begin{pmatrix} 0 \\ \Psi_+ \end{pmatrix} \right\rangle &= 0 \\
\left\langle \mathcal{A}\Phi_K \Leftrightarrow \alpha\Phi_K, \begin{pmatrix} 0 \\ \Psi_- \end{pmatrix} \right\rangle &= 0
\end{aligned} \tag{4.78}$$

The two components of $\mathcal{A}\Phi_K$ are

$$\begin{aligned}
(\mathcal{A}\Phi_K)_1 &= \sum_{s=-K-1, s \neq 0}^{K+1} (\lambda_s + i\kappa \Leftrightarrow \alpha) E^s U_s + \\
&\quad \Leftrightarrow i\kappa \sum_{t=-K, t \neq 0}^K F^t V_t \Leftrightarrow i\kappa G^+ \Psi_+ \Leftrightarrow i\kappa G^- \Psi_-, \\
(\mathcal{A}\Phi_K)_2 &= \Leftrightarrow i\kappa \sum_{s=-K-1, s \neq 0}^{K+1} E^s U_s + \sum_{t=-K, t \neq 0}^K (\tau_t + i\kappa \Leftrightarrow \alpha) F^t V_t + \\
&\quad + (i\mu + i\kappa \Leftrightarrow \alpha) G^+ \Psi_+ + (\Leftrightarrow i\mu + i\kappa \Leftrightarrow \alpha) G^- \Psi_-.
\end{aligned}$$

From (4.78) follows the system of equations

(I) For $s = \Leftrightarrow K \Leftrightarrow 1, \dots, K+1, s \neq 0$ and $t = \Leftrightarrow K, \dots, K, t \neq 0$ respectively

$$\begin{aligned}
0 &= \left[(\lambda_s + i\kappa \Leftrightarrow \alpha) E^s U_s \Leftrightarrow i\kappa \sum_{t=-K, t \neq 0}^K F^t V_t \Leftrightarrow i\kappa G^+ \Psi_+ \Leftrightarrow i\kappa \Psi_-, U_s \right]_1, \\
0 &= \left[\Leftrightarrow i\kappa \sum_{s=-K-1, s \neq 0}^{K+1} E^s U_s + (\tau_t + i\kappa \Leftrightarrow \alpha) F^t V_t, V_t \right]_2,
\end{aligned}$$

(II) and

$$\begin{aligned}
0 &= \left[\Leftrightarrow i\kappa \sum_{s=-K-1, s \neq 0}^{K+1} E^s U_s + (i\mu + i\kappa \Leftrightarrow \alpha) G^+ \Psi_+, \Psi_+ \right]_J, \\
0 &= \left[\Leftrightarrow i\kappa \sum_{s=-K-1, s \neq 0}^{K+1} E^s U_s + (\Leftrightarrow i\mu + i\kappa \Leftrightarrow \alpha) G^- \Psi_-, \Psi_- \right]_J.
\end{aligned}$$

After dividing by the norms and substituting the eigenvalue expansions from Section 4.2.4. we get

(I) For $s = \Leftrightarrow K \Leftrightarrow 1, \dots, K+1, s \neq 0$ and $t = \Leftrightarrow K, \dots, K, t \neq 0$ respectively

$$\begin{aligned}
0 &= \Leftrightarrow i\kappa \psi_+^s G^+ \Leftrightarrow i\kappa \psi_-^s G^- + (\lambda_s + i\kappa \Leftrightarrow \alpha) E^s \Leftrightarrow i\kappa \sum_{t=-K, t \neq 0}^K d_t^s F^t, \\
0 &= \Leftrightarrow i\kappa \sum_{s=-K-1, s \neq 0}^{K+1} c_s^t E^s + (\tau_t + i\kappa \Leftrightarrow \alpha) F^t.
\end{aligned}$$

(II) and

$$0 = (i\mu + i\chi \Leftrightarrow \alpha)G^+ \Leftrightarrow i\chi \sum_{s=-K-1, s \neq 0}^{K+1} b_s^+ E^s,$$

$$0 = (\Leftrightarrow i\mu + i\chi \Leftrightarrow \alpha)G^- \Leftrightarrow i\chi \sum_{s=-K-1, s \neq 0}^{K+1} b_s^- E^s,$$

Hence the condition on α to be an approximative eigenvalue of \mathcal{A} is that it be an eigenvalue of a quadratic complex matrix \mathcal{M} , of size $4K + 4$

$$\mathcal{M} := \begin{pmatrix} \mathcal{M}_{1,1} & \mathcal{M}_{1,2} & 0 \\ \mathcal{M}_{2,1} & \mathcal{M}_{2,2} & \mathcal{M}_{2,3} \\ 0 & \mathcal{M}_{3,1} & \mathcal{M}_{3,2} \end{pmatrix}$$

with some matrices $\mathcal{M}_{1,1}, \dots, \mathcal{M}_{3,2}$, listed in detail in Appendix C.2.1. \mathcal{M} acts on the vector

$$\left(G^+, G^-, E^{-K-1}, \dots, E^{-1}, E^1, \dots, E^{K+1}, F^{-K}, \dots, F^{-1}, \dots, F^{-1}, F^1, \dots, F^K \right)^t$$

The calculations of an estimate of $\Delta_K^{(j)}$ (see (4.76)) is given in Appendix C.2.2. The result is the uniform, but very rough, estimate

$$\Delta_K^2 \leq 3\chi^2 \Theta \left[\frac{28f(K)}{\rho^2 N} \left(\frac{\pi^2}{8} \Leftrightarrow g(K) \right) + \frac{8\pi^2}{|h_2|} \left(\frac{\pi^2}{8} \Leftrightarrow g(K+1) \right) \right] +$$

$$+ \frac{2\chi^2 \Theta}{\rho^2 N} \left[\left(\frac{\pi^2}{6} \Leftrightarrow f(K) \right) \left(8g(K+1) + 2f(K+1) \right) + \frac{2}{(2K+1)^2 \eta^2} + \frac{2}{(2K \Leftrightarrow 1)^2} \right]$$

with

$$\Theta = (h_1 \Leftrightarrow h_2)^2 N^4 \rho^4 / \pi^4, \quad f(K) = \sum_{t=1}^K \frac{1}{t^2}, \quad \text{and} \quad g(K) = \sum_{t=1}^K \frac{1}{(2t-1)^2}.$$

The limit for $K \rightarrow \infty$ of Δ_K is indeed zero. So for larger values of K one should get a good approximation at least of the approximate spectrum.

The mathematical software 'Matlab' provides a simple to use and still quite powerful function (called 'eig') for the numerical calculation of eigenvalues of a quadratic matrix with complex entries. Additionally it is possible to illustrate the resulting sets of values in graphs. However, it is rather slow, so that it is appropriate to use the programming language 'C' to calculate the entries of the large matrix \mathcal{M} . A pseudo code for the implementation, which is done in three steps, can be found in Appendix C.2.3.

4.4.3 Results of the numerical approximation

Some graphs produced by the programme can be found in the Appendix C.2.4. We plot the eigenvalues in different shapes in order see more clearly in the graph, where the respective eigenvalue lies. If we assume that the error of approximation is Δ_K then inside a disc of radius Δ_K around each calculated value lies an actual eigenvalue or spectral point of $\mathcal{A}(\chi)$. If the imaginary part of an $\alpha_K^{(j)}$ is greater than Δ_K this actual eigenvalue lies in the UHP. This value will be printed as '·'. If $\Delta_K > \text{Im}(\alpha_K^{(j)}) > 0$ then there is a chance that the actual eigenvalue lies in the LHP, the value will be

#	N	ρ	h_1	h_2	K	Δ_K
1	1	1	1	$\Leftrightarrow 1.1$	50	0.321151
2	1	1	1	$\Leftrightarrow 1.1$	200	0.160991
3	1	0.5	1	$\Leftrightarrow 1.1$	50	0.129016
4	1	0.5	1	$\Leftrightarrow 1.1$	200	0.064766
5	1	2	1	$\Leftrightarrow 1.1$	50	0.998734
6	2	0.5	0.01	$\Leftrightarrow 5.6$	50	0.919088
7	1	0.5	0.01	$\Leftrightarrow 1.1$	50	0.623757
8	1	1	0.01	$\Leftrightarrow 1.1$	50	1.287513
9	1	0.5	5	$\Leftrightarrow 7$	50	0.677997

Table 4.1: Δ_K for different sets of parameters, and for $\varkappa = 1$.

printed with a 'o'. A value with $0 < \text{Im}(\alpha_K^{(j)}) < \Leftrightarrow \Delta_K$ represents the chance, that the actual eigenvalue lies in the UHP, it will be printed with '*', whereas a value with $\Leftrightarrow \Delta_K < \text{Im}(\alpha_K^{(j)})$ stands for an actual eigenvalue certainly in the LHP. These values will be printed as 'x'.

The criteria for the choice of the parameters to run the programme are

- (a) $h_1 > 0$ and $h_2 < \Leftrightarrow 1/N$
- (b) The value of Δ_K should be small.
- (c) We want to verify that the hypothesis made in Section 4.3 is plausible. For that we need to choose the parameters such that

$$(i) \frac{h_1+1/N}{h_2+1/N} > 1 \quad , \text{ or } \quad (ii) \frac{h_1+1/N}{h_2+1/N} < 1$$

In Table 4.1 are shown a few examples of values of Δ_K for different choices of parameters N, ρ, h_1, h_2 and K when $\varkappa = 1$. Comparing the data in lines #1 and #2, also lines #3 and #4, we see that increasing the value of K by factor 4 will decrease the value of Δ_K only by factor 2. At the same moment, the amount of time needed to calculate the approximate values is increased by an enormous amount. Since the resulting pictures for $K = 200$ do not look much different to the ones for which $K = 50$, compare Fig. C.2 and C.3, we chose $K = 50$ to produce the graphs in Fig. C.3-C.6. The data in lines #1-#5 also show that Δ_K is smaller for smaller values of ρ , so the latter was set to $\rho = 0.5$ or $\rho = 1$ for the realisations of the programme. The sets of data in both lines #6 and #7 were chosen to yield the ratio $r := \frac{h_1+1/N}{h_2+1/N} = 0.1$, with different values of $h_1 > 0$ and $h_2 < \Leftrightarrow 1/N$. As the value of Δ_K is smaller for the data in line #7, this data was chosen to produce the graphs in Fig. C.5 and C.6. For $\rho = 1$, line #8, Δ_K is quite large again. For the parameters listed in line #9 the ratio is $r = 1$. At the same time the value of μ , the imaginary part of the complex eigenvalue of A_2 is quite large.

For $\varkappa = 0$ the spectra of $\mathcal{A}(0)$ for the different sets of parameters look exactly like the picture in the graph on the top of Fig. C.2, so for all other realisations $\varkappa = 0.05$ was chosen as the first value of \varkappa . One can see clearly that initially all eigenvalues go up, as it was shown by the series expansion of the eigenvalues of $\mathcal{A}(\varkappa)$ in Section 2.4.2. Note that the larger eigenvalues react faster (see especially Fig. C.3). We also observe that

the approximation of the spectrum obtained by the numerical approach looks similar to the ones obtained by the other approaches described in Sections 4.3.1 and 4.3.3.

The eigenvalue in the LHP, illustrated by the character 'x', does disappear very quickly for $r = 20$, which is the value of the mentioned ratio in Fig. C.3 and C.4 - stability in average is possible! These graphs also show that the choice of ρ does have an effect on that - for $\rho = 1$ in Fig. C.4 the eigenvalue in the LHP is lifted up for $\varkappa < 0.55$, for $\rho = 0.5$ (cf. Fig. C.3) a larger value of \varkappa is needed. This is probably due to the fact that then the value of μ is large. All graphs in Fig. C.4-C.6 have in common that the eigenvalue in the LHP is lifted up for values of \varkappa approximately equal to μ .

4.5 Summary

In Section 4.1 we described the solvable model of the wave equation on the finite string with the two different values $h = h_1 > 0$ and $h = h_2 < \frac{1}{N}$ in the boundary condition $u_x \Leftrightarrow hu|_0 = 0$ at $x = 0$. We defined what we meant by stability and showed that the positive energy form for the generator A_1 of the stable evolution is suitable to measure that. The energy form for the generator A_2 of the unstable dynamics is indefinite, but we could show that the positive inner product constructed by means of the canonical symmetry (cf. Appendix A.5) is also suitable for our purposes. However, the former self-adjoint operator A_2 now has a two-dimensional anti-hermitian part, it is not symmetric any more.

It was found in Section 4.2 that A_1 and A_2 are self-adjoint extensions of the same operator A_0 w.r.t. different inner products, and their resolvents differ only by a one-dimensional operator \hat{R}^α . We also observed that the operators A_1 and A_2 are approximately the same on a space of finite codimension. The consequence is that the mixing, the Markov chain results in, affects mainly the finite dimensional space spanned by the eigenvectors for eigenvalues with small absolute value. This observation motivated the numerical approximation by the Galerkin method.

The application of some results of Perturbation Theory did not prove successful. This was due to the fact that the values of \varkappa sufficient for stability certainly are not small, and that the norm of the operator $Q^t \times \text{id}_H$, if it exists, is not small either. Also are all eigenvalues of the unperturbed operator $\mathcal{A}(0)$ on the real axis. The theorems of Perturbation Theory give small discs around these values, in which the eigenvalues of the perturbed operator lie, however, they do not state, whether the eigenvalues lie in the upper half disc, which is required for stability.

Under the Assumption 4.6 made in Section 4.3 about the approximation of the spectrum of the non-symmetric operator \mathcal{A} we found by three different approaches (see Sections 4.3.1, 4.3.3 and 4.4) the same asymptotic behaviour of the spectrum near ∞ , illustrated in Figure 4.6. The approach using the resolvent also assumed that the norm of $\varkappa^2 \hat{R}^{\alpha-i\varkappa}$ is small.

The results of the numerical approximation, described in Section 4.4, suggest that it is in fact possible to choose \varkappa such that stability in average is obtained. The Hypothesis 4.5, however, is not really supported by the outcome so far. For both $r = 20$ and $r = 0.1$ it was the case that the eigenvalue in the LHP was lifted up, and in all cases it was for $\varkappa \approx \mu$. This was also the case for the example in Chapter 3, where \varkappa^2 needed to be greater or equal to $K_1^2 = \frac{r}{r-1}\omega_2^2$ - larger for larger imaginary part ω_2 of the eigenvalue in the LHP. However, the values of Δ_K are very large for larger \varkappa , so that the graphs

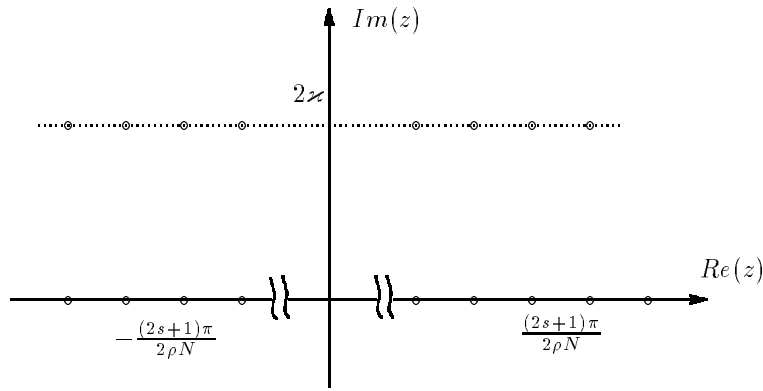


Figure 4.6: An asymptotic set for the spectrum $\sigma(\mathcal{A})$.

in Fig. C.5 and C.6 when $r = 0.1$ and $r = 1$, are not very reliable. Here one should really make the calculations for larger values of K .

One could continue the investigation and for example

- (i) consider the resolvent of \mathcal{A} , maybe apply Galerkin method to the operators T_1 and T_2 noted in Section 4.3.3,
- (ii) check whether \mathcal{A} is normal,
- (iii) make further use of the Frobenius-Schur factorisation,
- (iv) check the equivalence of the norms $\|\cdot\|_1$ and $\|\cdot\|_J$,
- (v) or consider the operator A_2 on the Krein space $(\mathcal{C}, [.,.]_2)$ and make use of the results of the Theory of Operators on Krein spaces.

Chapter 5

The wave equation on the semi-infinite string divided by a point mass and with random boundary conditions

The wave equation on the finite string with boundary condition $u|_N = 0$ studied in the previous chapter can be interpreted in another way as well: Imagine a semi-infinite string with a point mass M attached at $x = N$. But this mass is held tight so that it can not oscillate¹. Thus any displacement of the string to the left of the mass ($0 \leq x \leq N$) does not cause any oscillation of the string to the right of the mass ($N < x < \infty$) when it is released. Here we can ignore that part of the string.

In this chapter we study the model when the mass M can oscillate, it is not held (see Fig. 5.1). We are still interested solely in the evolution of the finite string inside the interval $[0, N]$, but now any wave that approaches to the mass M from the left will have a part transmitted to the right. The energy of that transmitted part is lost to the outside. We make use of the mathematical tools of the Lax/Phillips Scattering Theory (see Lax/Phillips[22]) to study this model, even though we do not deal with an actual

¹We could also say that the point mass has infinite mass.

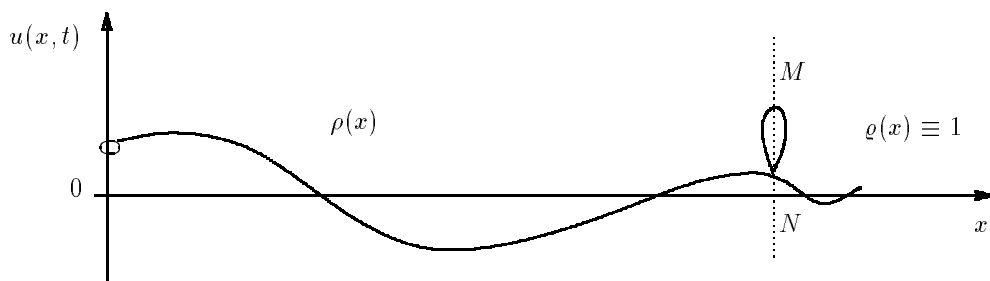


Figure 5.1: The semi-infinite string divided by a point mass.

scattering situation. To do this, we consider the wave equation on the semi-infinite string with some density function $\varrho(x)$. The part of the string for $N < x < \infty$ is not of actual interest, we set here $\varrho(x) \equiv 1$. The point mass M at $x = N$ is expressed by the Dirac-delta-function, so that we have for the density function

$$\varrho(x) = \rho(x)\chi_{[0,N)} + M\delta(x - N) + \chi_{(N,\infty)}.$$

For the solvable model we will set the tension $T \equiv 1$ and the density $\rho(x) = \rho^2$ constant. The boundary condition at $x = 0$ shall again be

$$u_x \Leftrightarrow h(t)u|_0 = 0$$

where $h(t)$ jumps between two values $h_1 > 0$ and $h_2 < 0$ via a symmetric 2-state Markov chain with infinitesimal generator Q . Since h_2 determines an unstable dynamics, we pursue the question whether there exist sufficient and/or necessary conditions for the choice of the intensity \varkappa of the Markov chain, so that the system is stable in average.

In the Section 5.1 are listed the results from the Lax/Phillips Scattering Theory which we use in the description of the solvable model in Section 5.2. However, in Section 5.3 we describe a different approach to the question of stability, not by means of random evolutions, but by directly considering the energy terms.

5.1 The Lax/Phillips scattering scheme applied to the wave equation on the semi-infinite string

Consider the wave equation on the positive half axis \mathbb{R}_0^+

$$\begin{cases} \varrho(x)u_{tt} = u_{xx} \\ u_x \Leftrightarrow hu|_0 = 0 \end{cases} \quad (5.1)$$

where the positive, bounded and continuous density function $\varrho(x)$ is equal to 1 for $x > N$, $N \in \mathbb{R}^+$. Define the differential operator $L = \Leftrightarrow \frac{1}{\varrho(x)} \frac{d^2}{dx^2}$, and the operator

$$A = i \begin{pmatrix} 0 & \Leftrightarrow 1 \\ L & 0 \end{pmatrix},$$

acting on the domain $\mathcal{D}(A)$ as a subspace of the space of Cauchy data

$$\mathcal{C} = \left\{ \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in \begin{pmatrix} W_1^2 \\ \mathcal{L}_2 \end{pmatrix} : (u_0)_x \Leftrightarrow hu_0|_0 = 0 \right\}$$

with energy form

$$\left[\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \right] = \frac{1}{2} \left(hu_0 \overline{v_0} + \int_0^\infty (u_0)_x \overline{(v_0)_x} + \varrho(x)u_1 \overline{v_1} dx \right)$$

just like for the case of a finite string in Chapter 4. We assume in this section that L is positive. Then A is self-adjoint and generates a one-parameter group of unitary operator e^{iAt} corresponding to the Cauchy problem

$$\begin{cases} \frac{1}{i} \frac{d}{dt} \mathcal{U} = A \mathcal{U} \\ \mathcal{U}|_{t=0} = (f, g)^t \end{cases} \quad (5.2)$$

A basic notion employed in the Lax/Phillips Scattering Theory is the one of the incoming and outgoing subspaces, see Lax/Phillips[22].

Definition 5.1

Let T be a one-parameter group of unitary operators on a Hilbert space \mathcal{C} . If there exist two closed subspaces \mathcal{D}_- and \mathcal{D}_+ of \mathcal{C} with the properties

- (i) $T(t)\mathcal{D}_- \subset \mathcal{D}_- \quad \forall t < 0, \quad T(t)\mathcal{D}_+ \subset \mathcal{D}_+ \quad \forall t > 0$
- (ii) $\bigcap_{t < 0} T(t)\mathcal{D}_- = \{0\}, \quad \bigcap_{t > 0} T(t)\mathcal{D}_+ = \{0\}$
- (iii) $\text{clos} \left(\bigcup_{t \in \mathbb{R}} T(t)\mathcal{D}_- \right) = \mathcal{C}, \quad \text{clos} \left(\bigcup_{t \in \mathbb{R}} T(t)\mathcal{D}_+ \right) = \mathcal{C}$

then \mathcal{D}_- and \mathcal{D}_+ are called **incoming** and **outgoing** subspaces respectively.

The solutions of (5.1) for $x > N$ have the form of d'Alembert waves (see Strauss[33])

$$u(x, t) = \Phi_1(x \Leftrightarrow t) + \Phi_2(x + t)$$

with some smooth functions Φ_1 and Φ_2 . $\Phi_1(x \Leftrightarrow t)$ is the wave moving to the right ('out') with $\frac{d}{dt}\Phi_1(x \Leftrightarrow t) = \Leftrightarrow \frac{d}{dx}\Phi_1(x \Leftrightarrow t)$, and $\Phi_2(x + t)$ is the wave moving to the left ('in') with $\frac{d}{dt}\Phi_2(x + t) = +\frac{d}{dx}\Phi_2(x + t)$. Thus possible choices for \mathcal{D}_+ and \mathcal{D}_- in our case are (see Pavlov[27])

$$\mathcal{D}_+^N = \left\{ \begin{pmatrix} u \\ -u_x \end{pmatrix} : u \in W_2^1, \text{supp}(u) \subset [N, \infty), u|_N = 0 \right\},$$

$$\mathcal{D}_-^N = \left\{ \begin{pmatrix} u \\ u_x \end{pmatrix} : u \in W_2^1, \text{supp}(u) \subset [N, \infty), u|_N = 0 \right\}.$$

Additional to above properties (i), (ii) and (iii), these two spaces are also orthogonal in the energy form since (note that $\varrho(x) \equiv 1$ for $x > N$)

$$\left[\begin{pmatrix} u \\ \Leftrightarrow u_x \end{pmatrix}, \begin{pmatrix} v \\ v_x \end{pmatrix} \right] = \frac{1}{2} \left(\int_N^\infty u_x \bar{v}_x + (\Leftrightarrow u_x) \bar{v}_x dx \right) = 0$$

The orthogonal complement of $\mathcal{D}_+^N \oplus \mathcal{D}_-^N$ in \mathcal{C} is the coinvariant subspace

$$\mathcal{K} = \left\{ \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \in \mathcal{C} : v_0 \equiv \text{const} \wedge v_1 \equiv 0 \text{ on } [N, \infty) \right\},$$

because for $\begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in \mathcal{K}$ it is

$$\left[\begin{pmatrix} v_0 \\ v_1 \end{pmatrix}, \begin{pmatrix} u \\ \pm u_x \end{pmatrix} \right] = \int_N^\infty (v_0)_x \bar{u}_x dx = 0.$$

Define $P_{\mathcal{K}}$ the orthogonal projection onto \mathcal{K} . The Cauchy data in \mathcal{K} has all the energy inside the interval $[0, N]$, the evolution of these data is exactly what we are interested in. The following theorem about the evolution on \mathcal{K} can be found in Lax/Phillips[22]

Theorem 5.2

Let \mathcal{D}_- and \mathcal{D}_+ be orthogonal incoming and outgoing subspaces for a group of unitary operators T on a Hilbert space \mathcal{C} , and $\mathcal{K} = \mathcal{C} \ominus (\mathcal{D}_+ \oplus \mathcal{D}_-)$. Then the operators

$$Z(t) = P_{\mathcal{K}}T(t)|_{\mathcal{K}}, \quad t \geq 0$$

form a C_0 -semi-group of contracting operators on \mathcal{K} with

- (i) $\lim_{t \rightarrow \infty} Z(t)f = 0 \quad \forall f \in \mathcal{K}$
(ii) $Z(t)\mathcal{D}_+ = 0 \quad Z(t)\mathcal{D}_- = 0$

In Pavlov[25] and Pavlov[27] are given some important facts for the study of the semi-group Z , especially for the case of the wave equation (5.1):

Definition 5.3

For $Im(k) > 0$ let $\varphi(x, k)$ be the function which satisfies

$$\Leftrightarrow \frac{d^2}{dx^2} \varphi(x, k) = k^2 \varrho(x) \varphi(x, k),$$

the boundary condition $\varphi' \Leftrightarrow h\varphi|_0 = 0$ and which is equal to $e^{ik(x-N)}$ for $x > N$, also called **Joost solutions**. If there exist a coefficient $S(k)$ such that

$$\varphi(x, k) + \overline{S(k)} \varphi(x, \Leftrightarrow k) \in \mathcal{L}_2[0, \infty)$$

then call $S(k)$ the **reflection coefficient**.

Lemma 5.4

For $Im(k) > 0$ exists a unique reflection coefficient $S(k)$. It is a bounded analytic function in the UHP with $|S(k)| < 1$. The limiting values

$$S(x) = \lim_{\varepsilon \rightarrow 0^+} S(x + i\varepsilon), \quad x \in \mathbb{R}$$

are almost everywhere unitary, i.e. $S(x)\overline{S(x)} = 1$ on \mathbb{R} , $S(k)$ is an inner function.

For the definition and some properties of inner functions refer to Appendix A.6. The characteristic function $\Theta_B(\xi)$ of a dissipative operator B is defined as (see Pavlov[26])

$$\Theta_B(\xi) = (\text{id} \Leftrightarrow CC^*)^{-1/2} (\text{id} \Leftrightarrow \xi C^*)^{-1} (\xi \text{id} \Leftrightarrow C) (\text{id} \Leftrightarrow C^* C)^{-1/2}$$

where the contraction $C = (B \Leftrightarrow i\text{id})(B + i\text{id})^{-1}$ is the Caley transform of B . For the spectrum of B holds the important result, to be found in the book “Harmonic Analysis of Operators on Hilbert Space” by Béla Sz.-Nagy and Ciprian Foias, see also Pavlov[25].

Theorem 5.5

The semi-group $Z(t) = P_{\mathcal{K}}T(t)|_{\mathcal{K}}$ is a semi-group of contractions. The characteristic function of its dissipative generator B is the reflection coefficient $S(k)$. The spectrum of B is the set of roots of $S(k)$ in the UHP.

SPECTRAL REPRESENTATION

From Lax/Phillips[22] we have the translation representation theorem

Theorem 5.6

Let T be a one-parameter group of unitary operators on a Hilbert space \mathcal{C} with incoming and outgoing subspaces \mathcal{D}_- and \mathcal{D}_+ . Then there exists a unitary map $\mathcal{T} : \mathcal{C} \rightarrow \mathcal{L}_2(\mathbb{R})$ such that

(1) The action of T on \mathcal{C} corresponds to the translation to the right by the variable $t \in \mathbb{R}$ in \mathcal{L}_2 .

(2) \mathcal{D}_- is mapped onto $\mathcal{L}_2(\mathbb{R}^-)$.

For the case of the wave equation it is shown in a paper by Ivanov and Pavlov [16] that this representation is given by the spectral representation \mathcal{T} (cf. Pavlov[27])

$$\mathcal{T} : \mathcal{U} = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \mapsto \frac{1}{\sqrt{2\pi}} \lim_{R \rightarrow \infty} \int_0^R (iku_0 + u_1) \overline{\varphi(x, k)} \varrho(x) dx =: \tilde{\mathcal{U}}(k)$$

with inverse mapping

$$\mathcal{T}^{-1} : \tilde{\mathcal{U}}(k) \mapsto \mathcal{U}(x) = \frac{1}{\sqrt{2\pi}} \lim_{R \rightarrow \infty} \int_{-R}^R \begin{pmatrix} 1/(ik) \\ 1 \end{pmatrix} \varphi(x, k) \tilde{\mathcal{U}}(k) dk$$

The mapping \mathcal{T} maps \mathcal{C} isometrically onto $\mathcal{L}_2(\mathbb{R})$, \mathcal{D}_-^N onto the Hardy space H_-^2 in the LHP (cf. Appendix A.6) and \mathcal{D}_+^N onto $S(k)H_+^2$, with $S(k)$ the reflection coefficient. Take for instance $\begin{pmatrix} u \\ -u_x \end{pmatrix} \in \mathcal{D}_+^N$, then with Definition 5.3

$$\begin{aligned} \mathcal{T} \begin{pmatrix} u \\ \Leftrightarrow u_x \end{pmatrix} &= \frac{1}{\sqrt{2\pi}} \lim_{R \rightarrow \infty} \int_N^R (iku \Leftrightarrow u_x) \left(e^{-ik(x-N)} + S(k)e^{ik(x-N)} \right) dx \\ &= \frac{1}{\sqrt{2\pi}} \lim_{R \rightarrow \infty} \left[\left[\Leftrightarrow u e^{-ik(x-N)} + u S(k) e^{ik(x-N)} \right]_{N+}^R \right. \\ &\quad \left. + \int_N^R u_x \left(e^{-ik(x-N)} \Leftrightarrow S(k) e^{ik(x-N)} \right) dx + \right. \\ &\quad \left. \Leftrightarrow \int_N^R u_x \left(e^{-ik(x-N)} + S(k) e^{ik(x-N)} \right) dx \right] \\ &= \Leftrightarrow \frac{2}{\sqrt{2\pi}} S(k) \int_N^\infty u_x e^{ik(x-N)} dx =: \tilde{\mathcal{U}}(k) \end{aligned}$$

With the Paley-Wiener Theorem A.62 follows

$$\int_N^\infty u_x e^{ik(x-N)} dx \in H_+^2$$

and thus $\tilde{\mathcal{U}}(k) \in S(k)H_+^2$, similarly for \mathcal{D}_-^N .

The action of e^{iAt} on \mathcal{C} is mapped onto the multiplication operator $(e^{ikt} *)$ on $\mathcal{L}_2(\mathbb{R})$, that is

$$\mathcal{T} e^{iAt} \mathcal{U} = e^{ikt} \tilde{\mathcal{U}}(k)$$

Consequently, the generator B of the contraction semi-group $Z(t)$ on \mathcal{K} is mapped onto the operator $P_K(k * \cdot)|_K$, where P_K is the projection onto $K = H_+^2 \ominus S(k)H_+^2$.

5.2 Description of the problem - a solvable model

Consider the solvable model for the wave equation

$$\begin{cases} \varrho(x) u_{tt} = u_{xx} \\ u_x \Leftrightarrow h u|_0 = 0 \end{cases} \quad (5.3)$$

now with

$$\varrho(x) = \rho^2 \chi_{[0,N)} + M\delta(x \Leftrightarrow N) + \chi_{[N,\infty)},$$

that is, a constant density function on the intervals $[0, N)$ and (N, ∞) and a point mass of mass M at $x = N$.

5.2.1 The point mass and the differential operator L

Let L be a differential operator such that the equation $\varrho(x)u_{tt} = u_{xx}$ can be written in the form $u_{tt} + Lu = 0$, but because of the Dirac delta function in $\varrho(x)$ it is now not possible to express the differential operator L as $\Leftrightarrow \frac{1}{\varrho(x)} \frac{d^2}{dx^2}$. However, it is possible to work with L as an operator, as described in the following.

The Dirac delta function can be included as a second condition on the functions of the domain of L . The solutions to (5.3) are continuous together with the first derivative in x (unless maybe at $x = N$) and the derivatives in t . Hence we have for $\varepsilon > 0$

$$\begin{aligned} & \int_{N-\varepsilon}^{N+\varepsilon} \varrho(x)u_{tt}dx = \int_{N-\varepsilon}^{N+\varepsilon} u_{xx}dx \\ \Leftrightarrow & \int_{N-\varepsilon}^N \rho^2 u_{tt}dx + \int_N^{N+\varepsilon} u_{tt}dx + Mu_{tt}(N) = u_x(N+\varepsilon) \Leftrightarrow u_x(N \Leftrightarrow \varepsilon) \end{aligned}$$

Let $\varepsilon \rightarrow 0$, then

$$Mu_{tt} = u_x(N+) \Leftrightarrow u_x(N \Leftrightarrow).$$

Thus the derivative at $x = N$ has a jump discontinuity, and if in addition u_{xx} is continuous on $[N, N+\varepsilon)$ for some $\varepsilon > 0$ it follows

$$Mu_{xx}(N+) = u_x(N+) \Leftrightarrow u_x(N \Leftrightarrow). \quad (5.4)$$

The domain $\mathcal{D}(L)$ then certainly includes the set

$$\left\{ y \in W_2^2 : y' \Leftrightarrow hy|_0 = 0, y''|_{N+} \text{ exists, and } My''|_{N+} = y'|_{N+} \Leftrightarrow y'|_{N-} \right\},$$

and here L acts as $\Leftrightarrow \frac{1}{\hat{\varrho}(x)} \frac{d^2}{dx^2}$ with $\hat{\varrho}(x) = \rho^2 \chi_{[0,N)} + \chi_{[N,\infty)}$.

The Dirac delta function $\delta(x \Leftrightarrow N)$ is often expressed by a sequence of functions (called *approximate identity*) $\eta_n(x \Leftrightarrow N)$ with the properties (see e.g. Hoffman[15])

- (i) $\lim_{n \rightarrow \infty} \eta_n(0) = \infty$,
- (ii) $\lim_{n \rightarrow \infty} \sup_{|x| > \varepsilon} (\eta_n(x)) = 0 \quad \forall \varepsilon > 0$,
- (iii) $\int_{-\infty}^{\infty} \eta_n(x)dx = 1 \quad \forall n \in \mathbb{N}$.

For such a sequence the density functions

$$\varrho_n(x) = \rho^2 \chi_{[0,N)} + M\eta_n(x \Leftrightarrow N) + \chi_{[N,\infty)}$$

are bounded, and the operators

$$L_n = \Leftrightarrow \frac{1}{\varrho_n(x)} \frac{d^2}{dx^2}$$

are well-defined and self-adjoint on their domains $\mathcal{D}(L_n)$ with inner products $\langle \cdot, \cdot \rangle_{\ell_n}$. The quadratic forms are

$$\langle L_n y, y \rangle_{\ell_n} = \Leftrightarrow \int_0^\infty y'' \bar{y} dx.$$

Now consider the operator L as the result of the limit of the quadratic forms $\langle L_n y, y \rangle_{\ell_n}$ for $n \rightarrow \infty$ and equip $\mathcal{D}(L)$ with the inner product $\langle \cdot, \cdot \rangle_\ell$, then again

$$\langle Ly, y \rangle_\ell = \Leftrightarrow \int_0^\infty y'' \bar{y} dx.$$

The spectral properties of L on $\mathcal{L}_{2,\ell}$ are the same as the Laplace operator $\Leftrightarrow \frac{d^2}{dx^2}$ on \mathcal{L}_2 , L is self-adjoint.

5.2.2 The reflection coefficient and the negative eigenvalue of the differential operator L

THE REFLECTION COEFFICIENT

The functions $e^{ik(x-N)}$ and $e^{-ik(x-N)}$ solve the differential equation $\Leftrightarrow \frac{d^2}{dx^2} y = k^2 \rho(x) y$ for $x > N$ for $k \in \mathbb{C}$. With view to the theory described in Section 5.1 the reflection coefficient $S(k)$ is of importance. It is determined in such a way that the functions $\varphi(x, k)$ (continuous and differentiable) with

$$\varphi(x, k) = e^{ik(x-N)} + \overline{S(k)} e^{-ik(x-N)}, \quad x > N$$

solve the boundary value problem on $[0, \infty)$

$$\begin{cases} \Leftrightarrow \frac{d^2}{dx^2} y = k^2 \rho(x) y \\ y' \Leftrightarrow h y|_0 = 0 \end{cases} \quad (5.5)$$

Divide the axis $[0, \infty)$ into two parts and require

$$\varphi(x, k) = \begin{cases} \varphi_-(x, k) & 0 \leq x < N \\ e^{ik(x-N)} + \overline{S(k)} e^{-ik(x-N)} & N < x \end{cases}$$

to satisfy the three properties

(i) $\varphi_-(x, k)$ solves (5.5) on $[0, N)$, i.e.

$$\begin{cases} \Leftrightarrow \frac{d^2}{dx^2} \varphi_-(x, k) = k^2 \rho^2 \varphi_-(x, k) \\ \varphi_-(0, k)' \Leftrightarrow h \varphi_-(0, k) = 0 \end{cases} \quad (5.6)$$

(ii) $\varphi(x, k)$ is continuous on $[0, \infty)$, i.e.

$$\varphi_-(N, k) = 1 + \overline{S(k)} \quad (5.7)$$

(iii) $\varphi'(x, k)$ has jump discontinuity at $x = N$ (cf. (5.4)), i.e.

$$\begin{aligned} M \varphi''(N+, k) &= \varphi'(N+, k) \Leftrightarrow \varphi'_-(N \Leftrightarrow, k) \\ \Leftrightarrow \Leftrightarrow k^2 M(1 + \overline{S(k)}) &= ik(1 \Leftrightarrow \overline{S(k)}) \Leftrightarrow \varphi'_-(N \Leftrightarrow, k). \end{aligned} \quad (5.8)$$

From (5.6) we get

$$\varphi_-(x, k) = \gamma e^{ik\rho x} + \gamma' e^{-ik\rho x}, \quad \gamma, \gamma' \in \mathbb{C},$$

with

$$(ik\rho\gamma \Leftrightarrow ik\rho\gamma') \Leftrightarrow h(\gamma + \gamma') = 0 \Leftrightarrow \gamma' = \frac{k\rho + ih}{k\rho \Leftrightarrow ih} \gamma.$$

We define

$$\vartheta := \frac{k\rho + ih}{k\rho \Leftrightarrow ih}$$

and get

$$\varphi_-(x, k) = \gamma (e^{ik\rho x} + \vartheta e^{-ik\rho x}), \quad \gamma \in \mathbb{C}. \quad (5.9)$$

Equations (5.7) and (5.8) yield the system

$$\begin{aligned} \gamma (e^{ik\rho N} + \vartheta e^{-ik\rho N}) &= 1 + \overline{S(k)} \\ \wedge \quad \rho ik\gamma (e^{ik\rho N} \Leftrightarrow \vartheta e^{-ik\rho N}) &= \overline{S(k)} (\Leftrightarrow ik + k^2 M) + ik + k^2 M, \end{aligned}$$

or, in matrix form,

$$\begin{pmatrix} e^{ik\rho N} + \vartheta e^{-ik\rho N} & \Leftrightarrow 1 \\ ik\rho (e^{ik\rho N} \Leftrightarrow \vartheta e^{-ik\rho N}) & ik \Leftrightarrow k^2 M \end{pmatrix} \begin{pmatrix} \gamma(k) \\ \overline{S(k)} \end{pmatrix} = \begin{pmatrix} 1 \\ ik + k^2 M \end{pmatrix}.$$

We apply Cramer's rule and get

$$\gamma(k) = \frac{(ik \Leftrightarrow k^2 M) + (ik + k^2 M)}{(ik \Leftrightarrow k^2 M) (e^{ik\rho N} + \vartheta e^{-ik\rho N}) + ik\rho (e^{ik\rho N} \Leftrightarrow \vartheta e^{-ik\rho N})},$$

and

$$\overline{S(k)} = \frac{(ik + k^2 M) (e^{ik\rho N} + \vartheta e^{-ik\rho N}) \Leftrightarrow ik\rho (e^{ik\rho N} \Leftrightarrow \vartheta e^{-ik\rho N})}{(ik \Leftrightarrow k^2 M) (e^{ik\rho N} + \vartheta e^{-ik\rho N}) + ik\rho (e^{ik\rho N} \Leftrightarrow \vartheta e^{-ik\rho N})}.$$

Simplifying we get for the reflection coefficient

$$S(k) = \frac{e^{ik\rho N} (1 + \rho + ikM) (k\rho \Leftrightarrow ih) + e^{-ik\rho N} (1 \Leftrightarrow \rho + ikM) (k\rho + ih)}{e^{ik\rho N} (1 \Leftrightarrow \rho \Leftrightarrow ikM) (k\rho \Leftrightarrow ih) + e^{-ik\rho N} (1 + \rho \Leftrightarrow ikM) (k\rho + ih)}. \quad (5.10)$$

Then we have for $k \in \mathbb{R}$

$$\overline{S(k)} = \frac{1}{S(k)}$$

so that $|S(k)| = 1$ for all $k \in \mathbb{R}$ as expected from Section 5.1, and we have $S(0) = \Leftrightarrow 1$.

The zeroes of $S(k)$ are determined as the solutions of the transcendental equation

$$\begin{aligned} 0 &= e^{ik\rho N} (1 + \rho + ikM) (k\rho \Leftrightarrow ih) + e^{-ik\rho N} (1 \Leftrightarrow \rho + ikM) (k\rho + ih) \\ \Leftrightarrow \quad e^{2ik\rho N} &= \frac{(\rho \Leftrightarrow 1 \Leftrightarrow ikM) (k\rho + ih)}{(\rho + 1 + ikM) (k\rho \Leftrightarrow ih)} \end{aligned} \quad (5.11)$$

Set $k = \alpha + i\beta$, $\alpha \in \mathbb{R}, \beta \geq 0$, then (5.11) is

$$e^{-2\beta\rho N} e^{2i\alpha\rho N} = \frac{[(\rho \Leftrightarrow 1 + \beta M) \Leftrightarrow i\alpha M] [(\alpha\rho) + i(\beta\rho + h)]}{[(\rho + 1 \Leftrightarrow \beta M) + i\alpha M] [(\alpha\rho) + i(\beta\rho \Leftrightarrow h)]} =: f(\alpha, \beta) \quad (5.12)$$

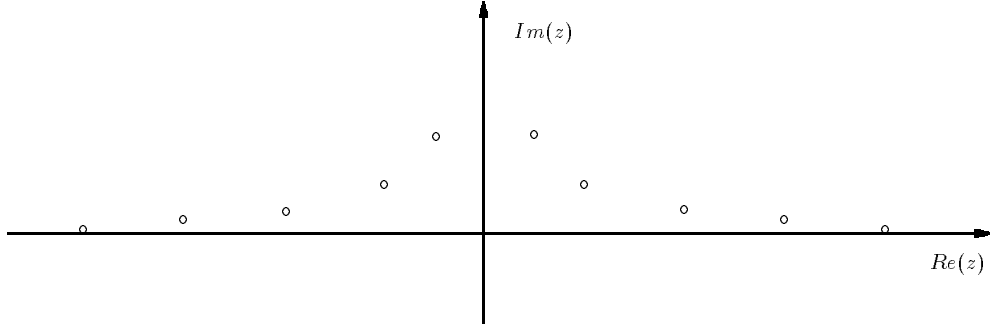


Figure 5.2: The zeroes of $S(k)$.

Assertion 5.7

The imaginary parts of the roots k_s of $S(k) = 0$ tend to zero for $|k_s| \rightarrow \infty$.

Proof:

Note first that since $S(k)$ is analytic its zeroes accumulate at most at infinity.

Assume $\exists \varepsilon > 0 : \forall \alpha_\varepsilon > 0 \exists z \in \mathbb{C} : [|\operatorname{Re}(z)| > \alpha_\varepsilon \wedge S(z) = 0 \wedge \operatorname{Im}(z) > \varepsilon]$.

Fix $\beta > 0$, then we get for the modulus of $f(\alpha, \beta)$ in (5.12)

$$|f(\alpha, \beta)|^2 = \frac{[(\rho \Leftrightarrow 1 + \beta M)^2 + \alpha^2 M^2][(\alpha \rho)^2 + (\beta \rho + h)^2]}{[(\rho + 1 \Leftrightarrow \beta M)^2 + \alpha^2 M^2][(\alpha \rho)^2 + (\beta \rho \Leftrightarrow h)^2]},$$

and

$$\lim_{|\alpha| \rightarrow \infty} |f(\alpha, \beta)| = 1.$$

But for the left hand side of (5.12) we have

$$\left| e^{-2\beta \rho N} e^{2i\alpha \rho N} \right| = e^{-2\beta \rho N} < 1 \quad \forall \alpha \in \mathbb{R},$$

and since for any root k_s it is necessary that $e^{-2\beta \rho N} = \left| f(\operatorname{Re}(k_s), \operatorname{Im}(k_s)) \right|$, the assumption is wrong. Hence

$$\forall \varepsilon > 0 : \exists \alpha_\varepsilon > 0 : \forall z \in \mathbb{C} : \left(|\operatorname{Re}(z)| > \alpha_\varepsilon \wedge \operatorname{Im}(z) > \varepsilon \Rightarrow S(z) \neq 0 \right).$$

□

Also, inserting $\Leftrightarrow k$ into the equation 5.10 yields $S(\Leftrightarrow k) = S(k)$, i.e. $S(k)$ is symmetric to the imaginary axis. Fig. 5.2 illustrates how the zeroes of $S(k)$ lie in the UHP.

THE NEGATIVE EIGENVALUE

Depending on the value of h in the boundary condition, the differential operator L has or has not one negative eigenvalue $\Leftrightarrow \mu^2, \mu > 0$. The six conditions on the corresponding eigenfunction $\psi(x, \mu)$ are

- (i) $\psi(x, \mu) \in \mathcal{L}_{2,\varrho}$

- (ii) $\psi(x, \mu)$ solves $\Leftrightarrow \frac{d^2}{dx^2} \psi(x, \mu) = \Leftrightarrow \mu^2 \psi(x, \mu)$ for $x > N$
- (iii) $\psi(x, \mu)$ solves $\Leftrightarrow \frac{d^2}{dx^2} \psi(x, \mu) = \Leftrightarrow \mu^2 \rho^2 \psi(x, \mu)$ for $0 \leq x < N$
- (iv) $\psi(x, \mu)$ fulfils the boundary condition $\psi'(0, \mu) \Leftrightarrow h \psi(0, \mu) = 0$
- (v) $\psi(x, \mu)$ is continuous.
- (vi) $\psi'(x, \mu)$ has jump discontinuity at $x = N$, i.e.

$$M\psi''(N+, \mu) = \psi'(N+, \mu) \Leftrightarrow \psi'(N \Leftrightarrow, \mu)$$

Conditions (i) and (ii) yield

$$\psi(x, \mu) = c_1 e^{-\mu(x-N)}, \quad x > N. \quad (5.13)$$

Conditions (iii) and (iv) give

$$\psi(x, \mu) = c_2 e^{\mu \rho x} + c_3 e^{-\mu \rho x}, \quad 0 \leq x < N \quad (5.14)$$

with

$$c_2(\rho \mu \Leftrightarrow h) \Leftrightarrow c_3(\rho \mu + h) = 0. \quad (5.15)$$

Finally, from conditions (v) and (vi) follows

$$\begin{aligned} c_1 &= c_2 e^{\mu \rho N} + c_3 e^{-\mu \rho N} \\ \mu^2 M c_1 &= \Leftrightarrow \mu c_1 \Leftrightarrow \mu \rho (c_2 e^{\mu \rho N} \Leftrightarrow c_3 e^{-\mu \rho N}). \end{aligned} \quad (5.16)$$

We consider three cases and recall that $\mu > 0$.

- (1) $\rho \mu \Leftrightarrow h = 0$, only possible for $h > 0$:
Then $c_3 = 0$ from (5.15) and (5.16) is

$$\begin{pmatrix} 1 & \Leftrightarrow e^{\mu \rho N} \\ \mu M + 1 & \rho e^{\mu \rho N} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with non-trivial solutions if and only if

$$\begin{aligned} 0 &= \rho e^{\mu \rho N} + e^{\mu \rho N}(\mu M + 1) \\ \Leftrightarrow 0 &= \rho + \mu M + 1. \end{aligned}$$

The system is not solvable since $\rho, M, \mu > 0$.

- (2) $\rho \mu + h = 0$, only possible for $h < 0$:
From (5.15) we get that $c_2 = 0$, and (5.16) is

$$\begin{pmatrix} 1 & \Leftrightarrow e^{-\mu \rho N} \\ \mu M + 1 & \Leftrightarrow \rho e^{-\mu \rho N} \end{pmatrix} \begin{pmatrix} c_1 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with non-trivial solutions if and only if

$$\begin{aligned} 0 &= e^{-\mu \rho N}(\Leftrightarrow \rho + \mu M + 1) \\ \Leftrightarrow \mu &= \frac{\rho \Leftrightarrow 1}{M}. \end{aligned}$$

Together with the assumption $\rho\mu + h = 0$ follows that $\Leftrightarrow\mu^2 = \Leftrightarrow\frac{(\rho-1)^2}{M^2}$ is an eigenvalue of L if and only if $\rho > 1$ and

$$\frac{\rho \Leftrightarrow 1}{M} = \Leftrightarrow\frac{h}{\rho} \Leftrightarrow h = \rho \frac{1 \Leftrightarrow \rho}{M} < 0.$$

Eigenfunction then is

$$\psi(x, \mu) = \begin{cases} e^{\mu\rho N} e^{-\mu\rho x} & , 0 \leq x < N \\ e^{-\mu\rho(x-N)} & , x \geq N \end{cases}$$

(3) $(\rho\mu \Leftrightarrow h)(\rho\mu + h) \neq 0$:

Then we get in (5.15)

$$c_3 = \frac{\rho\mu \Leftrightarrow h}{\rho\mu + h} c_2 =: \vartheta_\mu c_2$$

and from (5.16)

$$\begin{aligned} c_1 &= c_2 (e^{\mu\rho N} + \vartheta_\mu e^{-\mu\rho N}) \\ (\mu M + 1)c_1 &= \Leftrightarrow\rho c_2 (e^{\mu\rho N} \Leftrightarrow \vartheta_\mu e^{-\mu\rho N}) \end{aligned}$$

or

$$\begin{pmatrix} 1 & \Leftrightarrow(e^{\mu\rho N} + \vartheta_\mu e^{-\mu\rho N}) \\ (\mu M + 1) & \rho(e^{\mu\rho N} \Leftrightarrow \vartheta_\mu e^{-\mu\rho N}) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

with existing non-trivial solutions c_1 and c_2 if and only if

$$\begin{aligned} 0 &= \rho(e^{\mu\rho N} \Leftrightarrow \vartheta_\mu e^{-\mu\rho N}) + (\mu M + 1)(e^{\mu\rho N} + \vartheta_\mu e^{-\mu\rho N}) \\ \Leftrightarrow e^{2\rho\mu N} &= \frac{(\rho \Leftrightarrow 1 \Leftrightarrow \mu M)(\rho\mu \Leftrightarrow h)}{(\rho + 1 + \mu M)(\rho\mu + h)} =: f(\mu) \end{aligned} \quad (5.17)$$

Here we can consider the following two cases.

(a) For $h > 0$ there exist no solution to (5.17), since then for $\mu > 0$ and $\rho > 0$ either

(i) $f(\mu) < 0 < e^{2\rho\mu N}$,

(ii) or for $(\rho \Leftrightarrow 1 \Leftrightarrow \mu M) < 0 \wedge (\rho\mu \Leftrightarrow h) < 0$ it is

$$f(\mu) = \frac{(\Leftrightarrow\rho + 1 + \mu M)(\Leftrightarrow\rho\mu + h)}{(\rho + 1 + \mu M)(\rho\mu + h)} < 1 < e^{2\rho\mu N},$$

(iii) or for $(\rho \Leftrightarrow 1 \Leftrightarrow \mu M) > 0 \wedge (\rho\mu \Leftrightarrow h) > 0$ it is also

$$f(\mu) = \frac{(\rho \Leftrightarrow 1 \Leftrightarrow \mu M)(\rho\mu \Leftrightarrow h)}{(\rho + 1 + \mu M)(\rho\mu + h)} < 1 < e^{2\rho\mu N}.$$

(b) Exactly one solution exists, however, for $h < 0$. The function $f(\mu)$ has

(α) simple poles at $\mu = \Leftrightarrow h/\rho > 0$ and $\mu = (\Leftrightarrow 1 \Leftrightarrow \rho)/M < 0$

(β) simple zeroes at $\mu = h/\rho < 0$ and $\mu = (\rho \Leftrightarrow 1)/M$

(γ) at infinity the limit $\lim_{|\mu| \rightarrow \infty} f(\mu) = \Leftrightarrow 1$

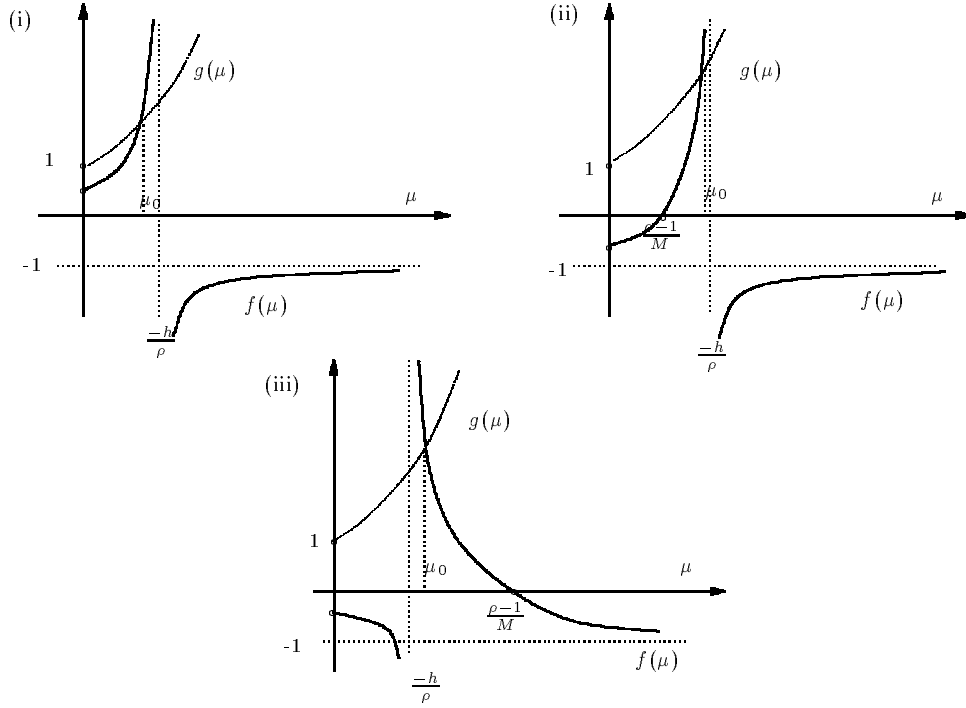


Figure 5.3: Sketches of the graphs of $f(\mu)$ and $g(\mu) = e^{2\mu\rho N}$.

(δ) at the origin the value $f(0) = (1 \Leftrightarrow \rho)/(1 + \rho) < 1$

In Figure 5.3 are shown sketches of the graphs of the function $f(\mu)$ and $g(\mu) = e^{2\mu\rho N}$ for $\mu > 0$ and

- (i) $0 < \rho \leq 1$,
- (ii) $1 < \rho$ and $\Leftrightarrow h\rho > (\rho \Leftrightarrow 1)/M$,
- (iii) $1 < \rho$ and $\Leftrightarrow h\rho < (\rho \Leftrightarrow 1)/M$.

The case $\Leftrightarrow h\rho = (\rho \Leftrightarrow 1)/M$ has been dealt with in part (2) above.

From above and from the sketches of the graphs we see that there exists exactly one negative eigenvalue $\Leftrightarrow \mu_0^2$ of L if and only if $h < 0$. The value μ_0 is near $\Leftrightarrow h_2/\rho$.

5.2.3 The stable system

Let $h = h_1 > 0$, there are no negative eigenvalues of the differential operator L_1 , the evolution $e^{iA_1 t}$ generated by the operator A_1 on \mathcal{C} is unitary w.r.t. the energy norm $\|\cdot\|_1$, i.e. stable (cf. Chapter 4). Apply the Lax/Phillips scattering scheme (cf. Section 5.1) to obtain the dissipative generator B_1 of the evolution

$$Z_1(t) = P_{\mathcal{K}} e^{iA_1 t} |_{\mathcal{K}}$$

on the space $\mathcal{K} = \mathcal{C} \ominus (\mathcal{D}_+^N \oplus \mathcal{D}_-^N)$. The reflection coefficient is, modified from (5.10),

$$S_1(k) = \frac{e^{ik\rho N}(1 + \rho + ikM) + \vartheta_1 e^{-ik\rho N}(1 \Leftrightarrow \rho + ikM)}{e^{ik\rho N}(1 \Leftrightarrow \rho \Leftrightarrow ikM) + \vartheta_1 e^{-ik\rho N}(1 + \rho \Leftrightarrow ikM)}$$

with

$$\vartheta_1 = \frac{k\rho + ih_1}{k\rho \Leftrightarrow ih_1}$$

The set of zeroes $\{\lambda_s : s \in \mathbb{N}\}$ of $S_1(k)$ in the UHP is the set of eigenvalues of B_1 , illustrated in Fig. 5.2. The spectral representation \mathcal{T}_1 w.r.t. the $\|\cdot\|_1$ -norm (cf. Section 5.1) maps the space \mathcal{K} to $K_1 = H_+^2 \oplus S_1 H_+^2$.

5.2.4 The unstable system

For $h = h_2 < 0$ there exist two complex eigenvalues $i\mu$ and $\Leftrightarrow i\mu$ of the generator A_2 of the evolution corresponding to the Cauchy problem (5.2). Define the eigenvectors $\Psi_+ = \begin{pmatrix} -\frac{1}{\mu}\psi(x,\mu) \\ \psi(x,\mu) \end{pmatrix}$ and $\Psi_- = \begin{pmatrix} \frac{1}{\mu}\psi(x,\mu) \\ \psi(x,\mu) \end{pmatrix}$ respectively. We decompose \mathcal{C} into the two invariant spaces $\Pi_- = \text{span}(\Psi_+, \Psi_-)$ and $\Pi_+ = \mathcal{C} \ominus \Pi_-$ with the corresponding projections P_- and P_+ and change the norm to $[\cdot, \cdot]_J$ with help of the canonical symmetry (cf. Section 4.1.4).

The incoming and outgoing subspaces are still the same as for $h = h_1$, i.e. \mathcal{D}_-^N and \mathcal{D}_+^N - the boundary condition does not effect these spaces. Our interest lies again in the dynamics on $\mathcal{K} = \mathcal{C} \ominus (\mathcal{D}_+^N \oplus \mathcal{D}_-^N)$. Though the complements are now taken w.r.t. the energy form $[\cdot, \cdot]_2$, the coinvariant spaces for both operators A_1 and A_2 are still the same, since the forms $[\cdot, \cdot]_1$ and $[\cdot, \cdot]_2$ coincide on all functions with support away from the origin (cf. Section 4.2.2).

MODIFIED SCATTERING THEORY

In Lax/Phillips[22] is described a modified scattering theory to deal with the case that the generator of the evolution does have complex eigenvalues. For each of the incoming and outgoing subspaces \mathcal{D}_- and \mathcal{D}_+ are defined two new subspaces

$$\begin{aligned} \mathcal{D}'_+ &= P_+ \mathcal{D}_+ & , & & \mathcal{D}''_+ &= \mathcal{D}_+ \cap \Pi_+ \\ \mathcal{D}'_- &= P_+ \mathcal{D}_- & , & & \mathcal{D}''_- &= \mathcal{D}_- \cap \Pi_+ \end{aligned}$$

and some scattering matrices are constructed in connection with these.

The eigenvectors Ψ_+, Ψ_- do not lie in $\mathcal{D}_+ = \mathcal{D}_+^N$ or in $\mathcal{D}_- = \mathcal{D}_-^N$, but the 'tails' with support in (N, ∞) do. Here it is $\psi(x, \mu) = e^{-\mu(x-N)}$ so that

$$\psi(x, \mu) = \frac{d}{dx} \left(\Leftrightarrow \frac{1}{\mu} \psi(x, \mu) \right).$$

It follows that the 'tails'

$$\chi_{(N, \infty)} \Psi_+ \in \mathcal{D}_- \quad \text{and} \quad \chi_{(N, \infty)} \Psi_- \in \mathcal{D}_+, \quad (5.18)$$

are the projections of Ψ_+ and Ψ_- onto \mathcal{D}_- and \mathcal{D}_+ respectively. The other parts $\chi_{[0, N)} \Psi_+$ and $\chi_{[0, N)} \Psi_-$ lie in \mathcal{K} and are eigenvectors of the evolution $(\cdot) = P_{\mathcal{K}} e^{iA_2 t} |_{\mathcal{K}}$ with the eigenvalues $i\mu$ and $\Leftrightarrow i\mu$. W.r.t. to the decomposition of \mathcal{C} the evolution is now decomposed into $Z_2(t) = Z(t)|_{\Pi_+}$ and $e^{iB_3 t}$ where B_3 is the square matrix

$$B_3 = i \begin{pmatrix} 0 & \Leftrightarrow 1 \\ \Leftrightarrow \mu^2 & 0 \end{pmatrix}$$

acting on

$$\Sigma : \text{span} \left(\chi_{[0,N]} \begin{pmatrix} \psi(x, \mu) \\ 0 \end{pmatrix}, \chi_{[0,N]} \begin{pmatrix} 0 \\ \psi(x, \mu) \end{pmatrix} \right) \subset \mathcal{K}.$$

In order to obtain the generator of $Z_2(t)$, apply the Lax/Phillips scattering scheme to the unitary evolution on Π_+ , generated by the self-adjoint operator $\hat{A}_2 := A_2|_{\Pi_+}$, with incoming and outgoing subspaces \mathcal{D}'_- and \mathcal{D}'_+ , which are orthogonal to Π_- . The result is a dissipative generator B_2 of the evolution $Z_2(t)$ with the reflection coefficient

$$S_2(k) = \frac{e^{ik\rho N}(1 + \rho + ikM) + \vartheta_2 e^{-ik\rho N}(1 \Leftrightarrow \rho + ikM)}{e^{ik\rho N}(1 \Leftrightarrow \rho \Leftrightarrow ikM) + \vartheta_2 e^{-ik\rho N}(1 + \rho \Leftrightarrow ikM)}$$

as characteristic function, where

$$\vartheta_2 = \frac{k\rho + ih_2}{k\rho \Leftrightarrow ih_2}.$$

We also have the spectral representation \mathcal{T}_2 , mapping $P_+\mathcal{K}$ onto $K_2 = H_+^2 \oplus S_2 H_+^2$. For the zeroes of $S_2(k)$ the same picture Fig 5.2 applies.

5.2.5 The expectation semi-group

Consider now the situation when the value of h jumps between $h_1 > 0$ and $h_2 < 0$ via a symmetric 2-state Markov chain with generator $Q = \varkappa \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$. Let \hat{P}_+ and \hat{P}_- be now the imbeddings of $P_+\mathcal{K}$ and Σ into \mathcal{K} respectively, and let id_+ and id_Σ be the identities on $P_+\mathcal{K}$ and Σ respectively. As it was done in Section 4.3, we express the generator $\mathcal{A}(\varkappa)$ of the expectation semi-group $\mathcal{E}(t)$ on the space $(\mathcal{K}, [., .]_1) \times (P_+\mathcal{K}, [., .]_2) \times (\Sigma, [., .]_J)$ as

$$\mathcal{A}(\varkappa) = \begin{pmatrix} B_1 + i\varkappa \text{id}_\mathcal{K} & \Leftrightarrow i\varkappa \hat{P}_+ & \Leftrightarrow i\varkappa \hat{P}_- \\ \Leftrightarrow i\varkappa P_+ & B_2 + i\varkappa \text{id}_+ & 0 \\ \Leftrightarrow i\varkappa P_- & 0 & B_3 + i\varkappa \text{id}_\Sigma \end{pmatrix}$$

Since the characteristic functions of B_1 and B_2 are known it is desirable to obtain a spectral representation of $\mathcal{A}(\varkappa)$ and its domain in order to study its spectral properties. However, for this the space Σ needs to be represented in the Hardy spaces.

The question how to do that has not been answered yet. The map \mathcal{T}_1 maps the vectors $\chi_{[0,N]}\Psi_+$ and $\chi_{[0,N]}\Psi_-$ onto some functions $f_+(k)$ and $f_-(k)$ in $H_+^2 \oplus S_1 H_+^2$, maybe one could 'add' these functions to K_2 . But what should one do then in the more general case of an n -state Markov chain? And how should one encode the eigenvalues $\pm i\mu$ into $\psi_\pm(k)$?

In the next Section another approach is described.

5.3 The energy approach to stability

It was seen in the previous Section that the evolution on the interval $[0, N]$ is contracting in the stable mode $h = h_1 > 0$ - the energy dissipates to the outside.

The unstable mode of evolution for $h = h_2 < 0$ is characterised by the fact that there exists the eigenvalue $e^{\mu t}$ of $\mathcal{E}(t)$, the energy in the corresponding one-dimensional subspace $Y_- = \text{span}(\chi_{[0,N]}\Psi_-)$ increases. However, the evolution on the complement

$Y_+ := \Pi_+ \oplus \text{span}(\chi_{[0,N]}\Psi_+)$ is contracting, the energy also dissipates.

When the Markov chain is activated and the dynamics jumps between the stable and the unstable mode, the system is stable in average precisely if the expected amount of energy dissipating to the outside exceeds the expected amount the energy increases by inside the interval.

Above observation suggests an approach to the question of stability in average in terms of the energy, different from the one based on the random evolution and the expectation semi-group.

Assume the evolution starts in the unstable mode with initial condition

$$\mathcal{U} = \mathcal{U}_0^+(0) + \alpha_0 \chi_{[0,N]}\Psi_-$$

where $\mathcal{U}_0^+ \in Y_+$. The energy contained in Y_- is

$$|\alpha_0| \|\chi_{[0,N]}\Psi_-\|$$

since $Y_- \perp Y_+$. After some time t_1 the system jumps into the stable mode, the evolution semi-group is now generated by B_1 and starts with initial condition

$$\mathcal{U}_1 = \mathcal{U}_0^+(t_1) + e^{\mu t_1} \alpha_0 \chi_{[0,N]}\Psi_-.$$

The eigenvectors Φ_s , $s \in \mathbb{N}$, for the eigenvalues k_s of B_1 form a basis for \mathcal{K} , w.r.t. which there exist unique coefficients $(\beta_1(s))_{s \in \mathbb{N}}$, such that

$$\mathcal{U}_1 = \sum_{s \in \mathbb{N}} \beta_1(s) \Phi_s.$$

The energy contained in Y_- is distributed among the eigenvectors of B_1 . When the system jumps back into the unstable mode after some time t_2 energy will have dissipated, and the vector

$$\mathcal{U}_2 = \sum_{s \in \mathbb{N}} e^{ik_s t_2} \beta_1(s) \Phi_s$$

is now expressed in the form

$$\mathcal{U}_2 = \mathcal{U}_2^+ + \alpha_1 \chi_{[0,N]}\Psi_-.$$

Above process is repeated.

One can expect that for certain pairs of times t_1 and t_2 one gets $|\alpha_1| \leq |\alpha_0|$. In order to establish above inequality one will need to determine how exactly the energies of the invariant spaces of the respective evolution semi-groups are distributed between each other. It seems plausible that a necessary condition for stability in average is that the expected value satisfies

$$E[|\alpha_1|] \leq |\alpha_0|.$$

To obtain above inequality the relations between the complex eigenvalue $\epsilon i \mu$ of A_2 , the imaginary parts of the eigenvalues k_s of B_1 , and the intensity \varkappa of the Markov process will need to fulfil certain conditions.

We saw in the last section, that the imaginary parts of the k_s tend to zero for large real parts. This means that mainly the energy dissipates which is contained in the

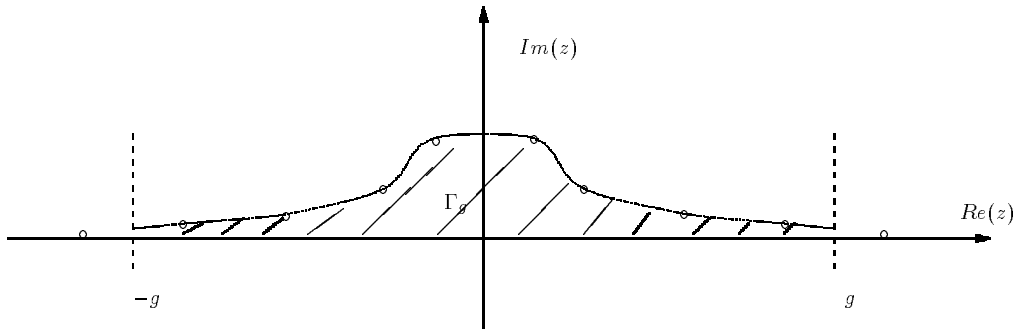


Figure 5.4: Illustration to Hypothesis 5.8.

subspace spanned by the eigenvectors for eigenvalues with small real part. Additionally, based on the observation made on the model in Chapter 4, we can make the conjecture that the energy on Y_- is mainly distributed onto exactly these eigenvectors, and vice versa. Say, we restrict our attention on the subspace

$$G_n := \text{span}_{|Re(k_s)| < g} \{ \Phi_s \} = \text{span} \{ \Phi_1, \dots, \Phi_n \}$$

for some $g > 0$. Then the rate of dissipation of the energy is given by the imaginary parts of k_1, \dots, k_n , and it can be estimated for instance by $\min \{ Im(k_s) : s = 1, \dots, n \}$. It is plausible that this rate should be large compared with μ , in order to obtain stability. This leads to the

Hypothesis 5.8

A necessary condition that stability in average is possible for the model investigated in this chapter is that the ratio of μ to either (i) $\min \{ Im(k_s) : |Re(k_s)| < g \}$ or (ii) $\sum_{|Re(k_s)| < g} Im(k_s)$ is small for some $g > 0$.

This condition can be interpreted geometrically, illustrated in Fig. 5.4, as follows: Interpolate the eigenvalues of B_1 and denote the area bounded by the resulting curve, the real axis and the lines $Re(z) = \pm g$ by Γ_g . Then the necessary condition mentioned in Hypothesis 5.8 reads $\mu, g \gg \mu$.

Chapter 6

Summary and further developments

In Chapter 1 we formulated the Conjecture 2.34 in the way that a sufficient and necessary condition for possible stability in average is that the operator $\hat{A} = \pi_1^2 A_1 + \pi_2^2 A_2$ generates a uniformly bounded semi-group, where A_1 and A_2 generate the two branches of the evolution of the system, and (π_1, π_2) is the equilibrium distribution of the Markov process.

THE CONCLUSION FROM CHAPTER 3

In the solvable model studied in Chapter 3 the operator $\pi_1^2 A_1 + \pi_2^2 A_2$ is

$$\hat{A} = \frac{1}{2} \begin{pmatrix} 0 & \Leftrightarrow i \\ i\omega_1^2 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & \Leftrightarrow i \\ \Leftrightarrow i\omega_2^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Leftrightarrow i \\ \frac{i}{2}(\omega_1^2 \Leftrightarrow \omega_2^2) & 0 \end{pmatrix}$$

with eigenvalues $\lambda_1 = \sqrt{\frac{1}{2}(\omega_1^2 \Leftrightarrow \omega_2^2)}$ and $\lambda_2 = \Leftrightarrow \sqrt{\frac{1}{2}(\omega_1^2 \Leftrightarrow \omega_2^2)}$, so that the conditions of Proposition A.30 for uniform boundedness is fulfilled if and only if $\omega_1^2 > \omega_2^2$ (\hat{A} is of Jordan form for $\omega_1^2 = \omega_2^2$). And by direct calculations we saw that stability in average is possible if and only if $\omega_1^2 > \omega_2^2$. Thus Conjecture 1.1 is supported by this simple example, and there is reason to believe that Conjecture 2.34 holds for all bounded operators. This will need further investigation.

THE CONCLUSION FROM CHAPTER 4

Although Hypothesis 4.5 made in Chapter 4 might not be correct, the study of the solvable model suggests that stability in average is possible.

Here it is not possible to check the conditions of Conjecture 2.34. The sum $\pi_1^2 A_1 + \pi_2^2 A_2$ is only defined on the common part $\mathcal{D}_0 = \mathcal{D}(A_1) \cap \mathcal{D}(A_2)$ of the domains of the operators and here it does not make any sense, since on \mathcal{D}_0 the operators coincide. Two modifications are by hand:

- (1) One could make use of the fact, that A_1 and A_2 are different self-adjoint extensions of the same operator A_0 defined on \mathcal{D}_0 . One would need to investigate how the semi-group $e^{iA_0 t}$, if A_0 is a generator, relates to $e^{iA_1 t}$ and $e^{iA_2 t}$, and how the choice of the norm plays a role. Certainly the question of equivalence of the norms is important.

- (2) The resolvents R_1^λ and R_2^λ of A_1 and A_2 are bounded operators, so it is possible to form the sum $\pi_1^2 R_1^\lambda + \pi_2^2 R_2^\lambda$, also are their spectra closely connected with the spectra of A_1 and A_2 respectively. Maybe the condition in Conjecture 1.1 can be modified to an expression in terms of the resolvents.

Notable is the observation made in Section 4.3.2 (see also Section 4.5) that the standard results of Perturbation Theory might not be applicable to show stability in average.

THE CONCLUSION FROM CHAPTER 5

The dissipative operators B_1 and $B_2 \oplus B_3$ again can not be added. If we assume they could we can apply the Trotter formula (Theorem A.31) to the exponential term in Conjecture 1.1 and get

$$\begin{aligned} \left\| e^{i(\pi_1^2 B_1 + \pi_2^2 (B_2 \oplus B_3))t} \right\| &= \left\| \lim_{n \rightarrow \infty} [e^{i\pi_1^2 B_1 t/n} e^{i\pi_2^2 (B_2 \oplus B_3) t/n}]^n \right\| \\ &\leq \lim_{n \rightarrow \infty} \left\| e^{i\pi_1^2 B_1 t/n} \right\|^n \left\| e^{i\pi_2^2 (B_2 \oplus B_3) t/n} \right\|^n. \end{aligned}$$

The term to the left is certainly less or equal to 1 if

$$\left\| e^{i\pi_1^2 B_1 t} \right\| \left\| e^{i\pi_2^2 (B_2 \oplus B_3) t} \right\| \leq 1 \quad \forall t \geq 0. \quad (6.1)$$

The operators $e^{i\pi_1^2 B_1 t}$ form a semi-group of contraction throughout the space \mathcal{K} , whereas $e^{i\pi_2^2 (B_2 \oplus B_3) t}$ are contracting only on $\mathcal{K} \ominus Y_-$ (cf. Section 5.3). Thus on $\mathcal{K} \ominus Y_-$ the inequality (6.1) is certainly fulfilled, and we need only check it on Y_- , that is

$$\left\| e^{i\pi_1^2 B_1 t} \chi_{[0, N]} \Psi_- \right\|_1 \leq e^{-\pi_2^2 \mu t} \left\| \chi_{[0, N]} \Psi_- \right\|_1 \quad \forall t \geq 0.$$

This inequality has a similar geometrical meaning to the condition stated in Hypothesis 5.8. One should compute the relation of the eigenvector $\chi_{[0, N]} \Psi_-$ to the eigenvectors Φ_s of B_1 and require that the energy of $\chi_{[0, N]} \Psi_-$ dissipates stronger under the action of $e^{i\pi_1^2 B_1 t}$ than it is increased by under the action of $e^{i\pi_2^2 B_3 t}$.

But since $\pi_1^2 B_1 + \pi_2^2 (B_2 \oplus B_3)$ is not defined, one will need to investigate how the resolvents, or in this particular case the reflection coefficients, can be put in relation to the evolution semi-groups.

Appendix A

Theory of Linear Operators and Inner Product Spaces

A.1 Linear operators on Hilbert space

Let H be a Hilbert space with the (positive) inner product $\langle \cdot, \cdot \rangle$ and the generated norm $\|\cdot\|$. For A and B linear operators with domains $\mathcal{D}(A)$ and $\mathcal{D}(B)$ we mean with $A \subset B$ that $\mathcal{D}(A) \subset \mathcal{D}(B)$ and $B|_{\mathcal{D}(A)} = A$.

The definitions listed in this section are standard and can be found in any book on linear operators on Hilbert spaces, for example Akhiezer/Glazman [1], Birman/Solomjak [2], Dowson [8], Hislop/Sigal [14], or Riesz/Sz.-Nagy [31].

Definition A.1

- (1) An operator A on H is said to be **bounded**, if there exists a constant $0 < M < \infty$ such that $\|Ax\| \leq M\|x\|$ for all $x \in \mathcal{D}(A)$. In this case, the infimum of all such constants is called the **operator norm** $\|A\|_{op}$ of A .
- (2) The operator A is said to be **closed** if following property holds:
Let $(x_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{D}(A)$ for which there exist $x, y \in H$ such that $\lim_{n \rightarrow \infty} x_n = x$ and $\lim_{n \rightarrow \infty} Ax_n = y$, then it follows that $x \in \mathcal{D}(A)$ and $Ax = y$.
- (3) If A maps any bounded set into a compact set, then A is called **compact** or **completely continuous**.

Clearly we have for an operator A

$$A \text{ compact} \Rightarrow A \text{ bounded} \Rightarrow A \text{ closed}$$

Definition A.2

Let A be an operator with dense domain $\mathcal{D}(A)$.

- (1) Set $\mathcal{D}^* = \{y \in H \mid \forall x \in \mathcal{D}(A) : \exists z \in H : \langle Ax, y \rangle = \langle x, z \rangle\}$ and define the **adjoint** A^* by $A^*y := z$ with domain $\mathcal{D}(A^*) = \mathcal{D}^*$.
- (2) If for all $x, y \in \mathcal{D}(A)$ we have $\langle Ax, y \rangle = \langle x, Ay \rangle$, then A is said to be **symmetric**, and it is $A \subset A^*$.

(3) If $A = A^*$, call A **self-adjoint**.

The proof of following lemma and theorem can be found for instance in Birman/Solomjak[2].

Lemma A.3

A self-adjoint operator is closed.

Theorem A.4

A symmetric operator A is self-adjoint if and only if the ranges of $A \Leftrightarrow i \text{ id}$ and $A + i \text{ id}$ are the whole space H .

Definition A.5

(1) $\lambda \in \mathbb{C}$ is called a **regular point** of A if the **resolvent** $R_\lambda = (A \Leftrightarrow \lambda \text{ id})^{-1}$ exists as a bounded operator defined on H . The set of all regular points, the **resolvent set**, is denoted by $\rho(A)$, its complement $\sigma(A) = \mathbb{C} \setminus \rho(A)$ is called the **spectrum**.

(2) The spectrum is divided into three disjoint parts (cf. Dowson[8]),

(i) the **point spectrum** or **discrete spectrum**

$$\sigma_d(A) = \{ \lambda \in \mathbb{C} \mid A \Leftrightarrow \lambda \text{ id is not one-to-one} \},$$

(ii) the **continuous spectrum**

$$\sigma_c(A) = \{ \lambda \in \mathbb{C} \mid A \Leftrightarrow \lambda \text{ id is one-to-one, but } \text{clos}[(A \Leftrightarrow \lambda \text{ id})\mathcal{D}(A)] = H \wedge (A \Leftrightarrow \lambda \text{ id})\mathcal{D}(A) \neq H \},$$

(iii) and the **residual spectrum**

$$\sigma_r(A) = \{ \lambda \in \mathbb{C} \mid A \Leftrightarrow \lambda \text{ id is one-to-one, but } \text{clos}[(A \Leftrightarrow \lambda \text{ id})\mathcal{D}(A)] \neq H \}$$

(3) The **approximate spectrum** is defined as

$$\sigma_a(A) = \{ \lambda \in \mathbb{C} \mid \exists (x_n) \in \mathcal{D}(A) : \|(A \Leftrightarrow \lambda \text{ id})x_n\| \rightarrow 0 \}$$

With the triangle inequality follows

Lemma A.6

If there exists a convergent sequence $(\lambda_n)_{n \in \mathbb{N}} \in \mathbb{C}$ with limit λ and a sequence $(x_n)_{n \in \mathbb{N}} \subset \mathcal{D}(A)$ such that

$$\lim_{n \rightarrow \infty} \frac{\|(A \Leftrightarrow \lambda_n \text{ id})x_n\|}{\|x_n\|} = 0,$$

then λ is in the approximative spectrum.

In Dowson[8] can be found

Theorem A.7

(1) $\sigma_d(A) \subset \sigma_a(A)$

(2) $\sigma_a(A)$ is a closed subset of the spectrum $\sigma(A)$.

See Kato[17] for more details about

Proposition A.8

The resolvent R_λ is an analytic, operator-valued function on $\varrho(A)$ with Taylor expansion at $\xi \in \varrho(A)$

$$R_\lambda = \sum_{n=0}^{\infty} (\lambda \Leftrightarrow \xi)^n R_\xi^{n+1} \quad \forall \lambda : |\lambda \Leftrightarrow \xi| < \|R_\xi\|^{-1}$$

Corollary A.9

For $\lambda \in \varrho(A)$ it is

$$\|R_\lambda\| \geq \frac{1}{\text{dist}(\lambda, \sigma(A))}.$$

The spectrum of a compact operator is characterised by the Riesz-Schauder theorem (see Hislop/Sigal[14])

Theorem A.10 (Riesz-Schauder)

The spectrum of a compact operator is discrete, the eigenvalues have finite multiplicity and accumulate at most at zero.

Corollary A.11

If the resolvent of an operator A is compact for some $\lambda \in \varrho(A)$, then the spectrum of A is discrete.

To proof the corollary observe that for $\mu \in \varrho(A)$ we have

$$\lambda \in \varrho(A) \Leftrightarrow \frac{1}{\lambda \Leftrightarrow \mu} \in \varrho(R_\mu).$$

The spectrum of a symmetric operator A is real, and simple calculations yield

Lemma A.12

For a symmetric operator A the norm of the resolvent R_λ with $\lambda \in \mathbb{C} \setminus \mathbb{R}$ is estimated by

$$\|R_\lambda\| \leq \frac{1}{|\text{Im}(\lambda)|}. \tag{A.1}$$

Important is

Theorem A.13 (Spectral Theorem)

If A is a self-adjoint operator then there exists a unique spectral family E_μ with

$$A = \int_{-\infty}^{\infty} \mu dE_\mu \tag{A.2}$$

For the proof of the theorem and the definition of the spectral family see for example Riesz/Sz.-Nagy[31], here the following remark is important.

Remark A.14

If A is self-adjoint has a discrete spectrum with eigenvalues λ_s and normalised eigenvectors v_s then the Spectral Theorem states that

$$\forall x \in \mathcal{D}(A) : Ax = \sum_s \lambda_s E_s x,$$

where $E_s = \langle \cdot, v_s \rangle v_s$ are the projections onto the subspaces spanned by the eigenvectors, and it says in particular that the eigenvectors form a complete set, in fact also orthogonal, for the Hilbert space H .

Corollary A.15

Let A be self-adjoint.

(1) If $\lambda \in \varrho(A)$, then we have

$$\|R_\lambda\| = \frac{1}{\text{dist}(\lambda, \sigma(A))}$$

(2) If $\mu \in \mathbb{C}$ and there exists $x \in \mathcal{D}(A)$ such that

$$\frac{\|(A \Leftrightarrow \mu \text{ id})x\|}{\|x\|} < \varepsilon,$$

then $\text{dist}(\mu, \sigma(A)) < \varepsilon$

For non-symmetric operators the distance to the numerical range can give an estimate of the norm of the resolvent (see Hislop[14]).

Definition A.16

The **numerical range** $\theta(A)$ of an operator A is the set

$$\theta(A) := \{ \langle Ax, x \rangle \mid x \in \mathcal{D}(A), \|x\| = 1 \} = \left\{ \frac{\langle Ax, x \rangle}{\|x\|^2} \mid x \in \mathcal{D}(A) \right\}$$

Certainly it is $\sigma_d(A) \subset \theta(A)$.

Proposition A.17

Let A be a closed operator and the range of $A \Leftrightarrow \lambda \text{ id}$ be dense in H for all $\lambda \in \mathbb{C} \setminus \text{clos}[\theta(A)]$. Then

(1) $\sigma(A) \subset \theta(A)$

(2) $\|R_\lambda\| \leq \left[\text{dist}(\lambda, \text{clos}[\theta(A)]) \right]^{-1}$

Corollary A.18

If there exist $\lambda \in \mathbb{C}$ and $u \in H$ such that $\|(A \Leftrightarrow \lambda \text{ id})u\| \leq \varepsilon \|u\|$ then $\text{dist}(\lambda, \text{clos}[\theta(A)]) \leq \varepsilon$.

A.2 One-parameter semi-groups of operators

The definitions and theorems listed in this section can be found in Pazy[29], and also Davies[6]. Let $(H, \langle \cdot, \cdot \rangle)$ be a Hilbert space.

Definition A.19

- (1) A family $T = \{T(t) : t \geq 0\}$ of bounded operators on a Hilbert space H (or Banach space) is called a **one-parameter semi-group of operators** if

$$(SG1) \quad T(0) = \text{id}_H$$

$$(SG2) \quad T(s+t) = T(s)T(t) = T(t)T(s) \quad \forall s, t \geq 0$$

- (2) A semi-group T is called **strongly continuous** or a C_0 -**semi-group** if

$$(SG3) \quad \lim_{t \rightarrow 0} \|T(t)x \ominus x\| = 0 \quad \forall x \in H$$

- (3) If the operators $T(t)$ for $t \geq 0$ are invertible, one can define $T(\Leftrightarrow t) := [T(t)]^{-1}$ and one obtains a **one-parameter group** of operators.

Theorem A.20

Let T be a C_0 -semi-group, then

- (1) there exist constants $M \geq 1$ and $\beta \geq 0$, such that $\|T(t)\| \leq Me^{\beta t}$ for all $t \geq 0$,
- (2) for all $x \in H$ the function $t \mapsto T(t)x$ is a continuous function from \mathbb{R}_0^+ to H .

Definition A.21

The **infinitesimal generator** A of a C_0 -semi-group T is the operator

$$A : x \mapsto Ax = \lim_{t \rightarrow 0} \frac{T(t)x \ominus x}{it}$$

with domain

$$\mathcal{D}(A) = \left\{ x \in H : \lim_{t \rightarrow 0} \frac{T(t)x \ominus x}{it} \text{ exists} \right\}.$$

We write $T(t) = e^{iAt}$.

Theorem A.22

A linear operator A is a generator of a C_0 -semi-group T with $\|T(t)\| \leq Me^{\beta t}$ if and only if

- (i) A is closed and $\mathcal{D}(A)$ is dense in H .
- (ii) If $\text{Im}(\lambda) < \Leftrightarrow \beta$, then $\lambda \in \varrho(A)$ and for all $n \in \mathbb{N}$ we have for the resolvent

$$\|R_\lambda^n\| \leq \frac{M}{|\text{Im}(\lambda) + \beta|^n}$$

Theorem A.23

Let T be a C_0 -semi-group and A its generator, then

(1) For $x \in H$ we have

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_t^{t+h} T(s)x ds = T(t)x$$

(2) For $x \in \mathcal{D}(A)$ it is $T(t)x \in \mathcal{D}(A)$ and

$$\frac{1}{i} \frac{d}{dt} T(t)x = AT(t)x = T(t)Ax$$

(3) For the point spectra of T and A holds

$$e^{i\sigma_d(A)t} \subset \sigma_d(T(t)) \subset e^{i\sigma_d(A)t} \cup \{0\}$$

Remark A.24

From above property (2) in Theorem A.23 follows that the vector $T(t)x$ solves the Cauchy problem

$$\begin{cases} \frac{1}{i} \frac{d}{dt} y(t) = A y(t) & t \geq 0 \\ y(0) = x \end{cases}$$

Important for the question of stability are the semi-groups of contractions and unitary operators.

Definition A.25

- (1) A **dissipative** operator A is one for which $\operatorname{Im}\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{D}(A)$ it is called **accretive**, if $\operatorname{Im}\langle Ax, x \rangle < 0$ for all $x \in \mathcal{D}(A)$.
- (2) A C_0 -semi-group T is called a semi-group of **contractions** if $\|T(t)\| \leq 1$ for all $t \geq 0$.
- (3) A C_0 -semi-group T with $\|T(t)\| = 1$ for all $t \geq 0$ is called a semi-group of **unitary** operators.
- (4) A C_0 -semi-group T is called **uniformly bounded** if there exists a constant $M \geq 1$ such that $\|T(t)\| \leq M$ for all $t \geq 0$.

Theorem A.26 (Lumer-Phillips)

Let A be a linear operator with dense domain $\mathcal{D}(A)$ on H , then A is the generator of a C_0 -semi-group of contractions if and only if

- (i) A is dissipative, and
- (ii) $\exists \mu < 0 : \operatorname{range}(i\mu \operatorname{id} \Leftrightarrow A) = H$

And for unitary semi-groups we have (see Davies[6])

Theorem A.27

The generators of semi-groups of unitary operators are precisely the self-adjoint operators .

Concerning uniformly bounded (or equi-bounded) semi-groups we find

Theorem A.28

A closed and densely defined operator A generates a uniformly bounded C_0 -semi-group T , satisfying $\|T(t)\| \leq M$ ($M \geq 1, t \geq 0$), if and only if $\mathbb{R}^+ \subset \varrho(A)$ and

$$\|R_{i\lambda}^n\| \leq \frac{M}{\lambda^n} \quad \text{for } \lambda > 0, n \in \mathbb{N}$$

Proposition A.29

Let A and B be two generators of C_0 -semi-groups of operators T and S resp. on a Hilbert (or Banach) space H , where B is dissipative. If A is similar to B , that is if there exists a bounded and invertible operator $X : H \rightarrow H$ with $Au = X B X^{-1} u \quad \forall u \in \mathcal{D}(A)$, then T is a uniformly bounded semi-group.

Proof:

Let $v \in \mathcal{D}(A)$, then with Remark A.24

$$u(t) = T(t)v \Leftrightarrow \frac{1}{i} \frac{d}{dt} u(t) = Au(t) \wedge u(0) = v$$

Now we define $w(t) = X S(t) X^{-1} v \in \mathcal{D}(A)$ and get

$$\begin{aligned} X^{-1} w(t) = S(t)(X^{-1} v) &\Leftrightarrow \frac{1}{i} \frac{d}{dt} (X^{-1} w(t)) = B(X^{-1} w(t)) \wedge X^{-1} w(0) = X^{-1} v \\ &\Leftrightarrow \frac{1}{i} \frac{d}{dt} w(t) = X B X^{-1} w(t) \wedge w(0) = v \\ &\Leftrightarrow \frac{1}{i} \frac{d}{dt} w(t) = A w(t) \wedge w(0) = v \end{aligned}$$

From the uniqueness of the solution of the Cauchy problem follows $w(t) = v(t)$. Since $v \in \mathcal{D}(A)$ was arbitrary, we get

$$e^{iAt} = X e^{iBt} X^{-1}.$$

Now it follows for all $t \geq 0$

$$\|e^{iAt}\| \leq \|X\| \|e^{iBt}\| \|X^{-1}\| \leq \|X\| \|X^{-1}\|$$

thus the operators $T(t) = e^{iAt}$ are uniformly bounded.

□

Proposition A.30

Let A be the generator of a C_0 -semi-group T on a finite dimensional Hilbert space H . Then T is uniformly bounded if and only if the following two conditions are satisfied:

- (i) All eigenvalues of A , for which the algebraic and geometric multiplicity are equal, have non-negative imaginary part.

- (ii) All eigenvalues of A , for which the algebraic multiplicity is larger than the geometric multiplicity, have positive imaginary part.

See Davies[6] for

Theorem A.31 (Trotter formula)

Let A, B be generators of C_0 -semi-groups on a Hilbert space, such that $Z = A + B$ exists and generates a semi-group. Then

$$e^{iZt} = \lim_{n \rightarrow \infty} \left(e^{iA \frac{t}{n}} e^{iB \frac{t}{n}} \right)^n$$

A.3 Perturbation Theory

Proposition A.32

Let A be a self-adjoint and V a bounded operator on a Hilbert space. For the spectra of A and $B = A + V$ holds

$$\text{dist}(\sigma(B), \sigma(A)) \leq \|V\|.$$

Proof:

For $\lambda \in \varrho(A) \cap \varrho(B)$ follows from $B = A + V$

$$(B \Leftrightarrow \lambda \text{ id})^{-1} \Leftrightarrow (A \Leftrightarrow \lambda \text{ id})^{-1} = (A \Leftrightarrow \lambda \text{ id})^{-1} V (A \Leftrightarrow \lambda \text{ id})^{-1},$$

and then

$$(B \Leftrightarrow \lambda \text{ id})^{-1} = [\text{id} + (A \Leftrightarrow \lambda \text{ id})^{-1} V]^{-1} (A \Leftrightarrow \lambda \text{ id})^{-1}. \tag{A.3}$$

From (A.3) we see that the resolvent of B exists whenever

$$\|(A \Leftrightarrow \lambda \text{ id})^{-1}\| \|V\| < 1 \Leftrightarrow \frac{1}{\|(A - \lambda \text{id})^{-1}\|} > \|V\|$$

With Corollary A.9 follows

$$\text{dist}(\lambda, \sigma(A)) > \|V\| \Rightarrow \lambda \in \varrho(A + V)$$

□

In Kato[17] one can find the definition and theorem below.

Definition A.33

Let A and V be two operators on a Hilbert space with $\mathcal{D}(A) \subset \mathcal{D}(V)$.

V is called **relatively bounded w.r.t. A** , if

$$\exists a, b > 0 : \forall x \in \mathcal{D}(A) : \|Vx\| \leq a\|x\| + b\|Ax\| \tag{A.4}$$

Theorem A.34

Let A be a closed operator on H and V relatively bounded w.r.t. A with constants a and b as in (A.4). If there is $\lambda \in \varrho(A)$ such that

$$a\|(A \Leftrightarrow \lambda \text{ id})^{-1}\| + b\|A(A \Leftrightarrow \lambda \text{ id})^{-1}\| < 1 \tag{A.5}$$

then $B = A + V$ is closed and $\lambda \in \rho(B)$ with

$$\|(B \Leftrightarrow \lambda \text{id})^{-1}\| \leq \|(A \Leftrightarrow \lambda \text{id})^{-1}\| (1 \Leftrightarrow a \|(A \Leftrightarrow \lambda \text{id})^{-1}\| \Leftrightarrow b \|A(A \Leftrightarrow \lambda \text{id})^{-1}\|)^{-1}$$

Additionally, if A has compact resolvent so has B .

Corollary A.35

If a self-adjoint operator A has compact resolvent, i.e. discrete spectrum, then any perturbation $B = A + V$ by a bounded operator V has discrete spectrum as well.

Proof:

If V is bounded then it is relatively bounded w.r.t. A with constants $a = \|V\|$ and $b = 0$. Choose λ with $\text{Im}(\lambda) > a > 0$, then we get from (A.1)

$$\|(A \Leftrightarrow \lambda \text{id})^{-1}\| < \frac{1}{\text{Im}(\lambda)} < \frac{1}{a}$$

so that (A.5) is fulfilled and above theorem applies, since A self-adjoint implies that it is closed. Thus B has compact resolvent, and its spectrum is discrete (cf. Corollary A.11).

□

The proof of the next theorem is given in Pazy[29].

Theorem A.36

Let A be the infinitesimal generator of a C_0 -semi-group T on a Hilbert space H with $\|T(t)\| \leq Me^{\beta t}$. If V is a bounded linear operator on H then $A + V$ generates a C_0 -semi-group S on H with

$$\|S(t)\| \leq Me^{(\beta + M\|V\|)t}.$$

For further results of Perturbation Theory see for example the book by T. Kato ([17]).

A.4 A differential operator on $\mathcal{L}_2[0, N]$

Let $\mathcal{L}_2(X)$ be the space of all square-summable complex-valued functions on an interval X with the inner product $\langle \cdot, \cdot \rangle$

$$\langle y, z \rangle = \int_X y \bar{z} dx$$

Denote by $\mathcal{L}_{2,\varrho}(X)$ the space with the weighted inner product

$$\langle y, z \rangle_\varrho = \int_X \varrho y \bar{z} dx$$

with the real, continuous and positive weight function $\varrho(x)$.

From Ziemer[36] we have the following definition and theorem.

Definition A.37

The **Sobolev spaces** $W_2^l(\mathbb{R})$ for $l \in \mathbb{N}_0$ are defined as

$$W_2^l(\mathbb{R}) = \left\{ u \in \mathcal{L}_2(\mathbb{R}) \mid \forall 0 \leq p \leq l : D^p u = \frac{d^p}{dx^p} u \text{ exists } \wedge D^p u \in \mathcal{L}_2(\mathbb{R}) \right\}.$$

The Sobolev space $W_2^l(\mathbb{R})$ is a Banach space with the norm

$$\|u\|_{W_2^l} = \left(\int_{\mathbb{R}} \sum_{p=0}^l |D^p u|^2 dx \right)^{\frac{1}{2}}.$$

Theorem A.38

(1) $C_0^\infty(\mathbb{R})$ is dense in $W_2^l(\mathbb{R})$ w.r.t. the W_2^l -norm, that is $\text{clos}_{W_2^l}(C_0^\infty(\mathbb{R})) = W_2^l$

(2) And also $\text{clos}_{W_2^{l-1}}(W_2^l) = W_2^{l-1}$

For example in Naimark[24] one can find the

Definition A.39

(1) A **linear differential expression** of order $n = 2$ is an expression of the form

$$l(y) = p_0(x)y''(x) + p_1(x)y'(x) + p_2(x)y(x)$$

with functions $p_2(x), p_1(x)$ and $p_0(x)$.

(2) A set of **boundary conditions** is a collection of linear forms $\{U_j\}$ in the variables $y(0), y'(0), y(N), y'(N)$, e.g.

$$\begin{aligned} U_1(y) &= \alpha_1 y(0) + \alpha_2 y'(0) + \alpha_3 y(N) + \alpha_4 y'(N) \\ U_2(y) &= \beta_1 y(0) + \beta_2 y'(0) + \beta_3 y(N) + \beta_4 y'(N) \end{aligned}$$

together with the conditions $U_j(y) = 0$. At least one of the α_j 's and at least one of the β_j 's should be non-zero.

(3) A **differential operator** L is a pair $(l, \mathcal{D}(L))$, where l is a differential expression and $\mathcal{D}(L)$, the domain of L , is the set of all functions l can be applied to and which satisfy certain boundary conditions $U_j(y) = 0$.

In Tikhonov[35] can be found following the definition and theorem.

Definition A.40

The **Green's function** for a **Sturm-Liouville operator** L on $\mathcal{L}_2[0, N]$ with differential expression

$$l(y) = \frac{d}{dx} \left(p(x) \frac{d}{dx} y(x) \right) + q(x)y(x)$$

is a function $G(x, \xi)$ with the properties

(i) For $0 < x < \xi$ and $\xi < x < N$ it is $l(G(\cdot, \xi)) = 0$.

- (ii) $G(., \xi)$ satisfies the boundary conditions in x .
- (iii) $G(., \xi)$ is continuous on $[0, N]$.
- (iv) The first derivative exists with one jump discontinuity at $x = \xi$, and

$$\frac{d}{dx}G(x, \xi) \Big|_{x=\xi+0} \Leftrightarrow \frac{d}{dx}G(x, \xi) \Big|_{x=\xi-0} = \frac{1}{p(x)}$$

Theorem A.41

Assume the homogeneous boundary-value problem $L(y) = 0, y \in \mathcal{D}(L)$, possesses only the trivial solution, then the solution of the inhomogeneous boundary-value problem $L(y) = f$ is expressed by

$$y(x) = \int_0^N G(x, \xi) f(\xi) d\xi = \langle G(x, .), \bar{f} \rangle$$

Definition A.42

The **Wronskian** of two functions $y, z \in C^1(\mathbb{R})$ is the determinant

$$\Delta(x) := \begin{vmatrix} y(x) & z(x) \\ y'(x) & z'(x) \end{vmatrix},$$

also denoted by $W(y, z)|_x$.

With regard to the problems described in Chapters 4 and 5 we consider here only the special differential operator L with differential expression

$$l(y) = \Leftrightarrow \frac{1}{\varrho(x)} \frac{d^2}{dx^2} \tag{A.6}$$

on the domain $\mathcal{D}(L)$ defined by the so called Sturm-Liouville boundary conditions

$$\begin{cases} \alpha_1 y(0) + \alpha_2 y'(0) = 0 \\ \beta_1 y(N) + \beta_2 y'(N) = 0 \end{cases} \tag{A.7}$$

Immediately from

$$\begin{aligned} \frac{d}{dx}W(y, z) &= \frac{d}{dx}(yz' \Leftrightarrow y'z) = yz'' \Leftrightarrow y''z \\ &= y[\Leftrightarrow \lambda \varrho z] \Leftrightarrow [\Leftrightarrow \lambda \varrho y]z = 0 \end{aligned}$$

follows

Lemma A.43

The Wronskian of two solutions y and z of

$$l(y) = \Leftrightarrow \frac{1}{\varrho(x)} \frac{d^2}{dx^2} y \Leftrightarrow \lambda y = 0 \quad , \lambda \in \mathbb{C}$$

is constant.

Remark A.44

The differential expression $l = \Leftrightarrow \frac{1}{\varrho(x)} \frac{d^2}{dx^2}$ on the space $\mathcal{L}_{2,\varrho}(X)$ has analogous properties to the much simpler expression $l_0 = \Leftrightarrow \frac{d^2}{dx^2}$ on $\mathcal{L}_2(X)$ with corresponding differential operator L_0 , which is of the form of a Sturm-Liouville operator. Many equations connected with l can easily be modified to equations with l_0 like

$$l(y) = f \quad \Leftrightarrow \quad l_0(y) = \varrho f$$

The formulae for l_0 on $\mathcal{L}_2(X)$ hold for l on $\mathcal{L}_{2,\varrho}(X)$. This is of particular use if $1/\varrho(x)$ is not differentiable or has singularities.

The differential operator L is symmetric on the space $\mathcal{L}_{2,\varrho}[0, N]$, since for $y, z \in \mathcal{D}(L)$ we have (integrating by parts)

$$\begin{aligned} \int_0^N \varrho l(y) \bar{z} dx &= [\Leftrightarrow y' \bar{z}]_0^N + \int_0^N y' \bar{z}' dx \\ &= [\Leftrightarrow y' \bar{z} + y \bar{z}']_0^N + \int_0^N \varrho y \bar{l}(z) dx \end{aligned} \quad (\text{A.8})$$

So, in particular, its eigenfunctions are orthogonal to each other, in fact they form a complete set, since also

Proposition A.45

L is self-adjoint on $\mathcal{L}_{2,\varrho}[0, N]$

Proof:

We need to show that $\mathcal{D}(L^*) \subset \mathcal{D}(L)$, as from (A.8) follows that the differential expression for L^* is also l (cf. Naimark[24]). So let $y \in \mathcal{D}(L)$, we want to find all $z \in C^2$ such that

$$\langle L(y), z \rangle_\varrho = \langle y, l(z) \rangle_\varrho.$$

From (A.8) we get the condition

$$[\Leftrightarrow y' \bar{z} + y \bar{z}']_0^N = 0$$

W.l.o.g. assume that in (A.7) $\alpha_2 \neq 0$ and $\beta_2 \neq 0$., then

$$\begin{aligned} 0 &= [\Leftrightarrow y' \bar{z} + y \bar{z}']_0^N \\ \Leftrightarrow \quad 0 &= \left(\frac{\alpha_1}{\alpha_2} \bar{z} + \bar{z}' \right) y(N) \Leftrightarrow \left(\frac{\beta_1}{\beta_2} \bar{z} + \bar{z}' \right) y(0) \end{aligned}$$

This must hold for all $y \in \mathcal{D}(L)$, it follows

$$\begin{aligned} \langle L(y), z \rangle_\varrho = \langle y, l(z) \rangle_\varrho &\Leftrightarrow \left(\frac{\alpha_1}{\alpha_2} \bar{z} + \bar{z}' \right) = 0 \wedge \left(\frac{\beta_1}{\beta_2} \bar{z} + \bar{z}' \right) \\ &\Leftrightarrow z \in \mathcal{D}(L) \end{aligned}$$

□

THE RESOLVENT $(L \Leftrightarrow \lambda)^{-1}$

We follow the procedure described for instance in Tikhonov[35] to construct the

Green's function of the operator $L_0 \Leftrightarrow \lambda \varrho$, $\lambda \in \mathbb{C}$, on $[0, N]$, this will yield the resolvent $(L \Leftrightarrow \lambda)^{-1}$.

1. Let $\varphi(x)$ and $\vartheta(x)$ be solutions to

$$\Leftrightarrow \frac{d^2}{dx^2} y \Leftrightarrow \lambda \varrho y = 0 \quad (\text{A.9})$$

which satisfy the boundary condition at $x = 0$ and at $x = N$ respectively.

2. The Green's function is given as

$$G(x, \xi) = \frac{1}{W(\varphi, \vartheta)} \begin{cases} \varphi(x)\vartheta(\xi) & 0 \leq x \leq \xi \\ \varphi(\xi)\vartheta(x) & \xi \leq x \leq N \end{cases} \quad (\text{A.10})$$

3. The solution y of the boundary value problem

$$\begin{cases} \Leftrightarrow \frac{d^2}{dx^2} y \Leftrightarrow \lambda \varrho y = f \\ \alpha_1 y(0) + \alpha_2 y'(0) = 0 \\ \beta_1 y(N) + \beta_2 y'(N) = 0 \end{cases}$$

is then given by the integral expression

$$y(x) = \int_0^N G(x, \xi) f(\xi) d\xi.$$

4. To obtain the resolvent $(L \Leftrightarrow \lambda)^{-1}$ of L , we observe that

$$\begin{aligned} \Leftrightarrow \frac{1}{\varrho} \frac{d^2}{dx^2} y \Leftrightarrow \lambda y &= f \\ \Leftrightarrow \Leftrightarrow \frac{d^2}{dx^2} y \Leftrightarrow \lambda \varrho y &= \varrho f \end{aligned}$$

Thus $(L \Leftrightarrow \lambda)^{-1}$ is the integral operator

$$(L \Leftrightarrow \lambda)^{-1} = K^\lambda : f(\cdot) \mapsto \int_0^N G(\cdot, \xi) \varrho(\xi) f(\xi) d\xi \quad (\text{A.11})$$

Remark A.46

For bounded $\varrho(x)$ we have

$$\int_0^N \int_0^N |G(x, \xi)|^2 \varrho(x) \varrho(\xi) dx d\xi < \infty,$$

so that K^λ is of Hilbert-Schmidt type, thus compact (cf. Hislop/Sigal [14]), L has discrete spectrum (Riesz-Schauder Theorem A.10).

A.5 Linear spaces with an indefinite metric

The definitions and propositions listed in this section can be found in the book by I.S. Iohvidov, M.G. Krein and H. Langer[21].

KREIN SPACE

Definition A.47

Let \mathcal{K} be a linear space.

- (1) A **bilinear metric** or **inner product** on \mathcal{K} is a mapping $\mathcal{K} \times \mathcal{K} \rightarrow \mathbb{C}$ with the properties

- (i) $[x, y] = \overline{[y, x]}$
(ii) $[\lambda x + \mu y, z] = \lambda[x, z] + \mu[y, z]$

for $\lambda, \mu \in \mathbb{C}$ and $x, y, z \in \mathcal{K}$.

- (2) An inner product is called an **indefinite metric**, if there exist $x, y \in \mathcal{K}$ such that $[x, x] > 0$ and $[y, y] < 0$.

- (3) A sub-manifold $L \subset \mathcal{K}$ is called

positive, if $\forall x \in L : [x, x] > 0$,
neutral, if $\forall x \in L : [x, x] = 0$,
negative, if $\forall x \in L : [x, x] < 0$.

Proposition A.48

For an indefinite metric there exist neutral elements.

Proof:

Let $x, y \in \mathcal{K}$ with $[x, x] > 0$ and $[y, y] < 0$.

The mapping

$$T : \lambda \mapsto [\lambda x + (1 \Leftrightarrow \lambda)y, \lambda x + (1 \Leftrightarrow \lambda)y]$$

is continuous with $T(0) > 0$ and $T(1) < 0$. Thus there is $0 \leq \lambda_0 \leq 1$ such that $T(\lambda_0) = 0$ and $\lambda_0 x + (1 \Leftrightarrow \lambda_0)y$ is a neutral element.

□

The notion of orthogonality exists also for spaces with an indefinite metric.

Definition A.49

- (1) $x, y \in \mathcal{K}$ are said to be **orthogonal**, written $x[\perp]y$, if $[x, y] = 0$.
(2) The **orthogonal complement** of a sub-manifold $M \subset \mathcal{K}$ is

$$M^{[\perp]} = \{x \in \mathcal{K} : x[\perp]M\}$$

- (3) $\mathcal{K}^0 = \mathcal{K} \cap \mathcal{K}^{\perp}$ is called the **isotrope part** of \mathcal{K} . If $\mathcal{K}^0 = \{0\}$, then \mathcal{K} is called **non-degenerate**.

The direct sum $L[\oplus]M$ of two sub-manifolds as well as the orthogonal projections are defined just like for Hilbert spaces with positive inner products.

Definition A.50

- (1) A space $(\mathcal{K}, [., .])$ with indefinite metric is called a **Krein space**, if there exists subspaces \mathcal{K}_+ and \mathcal{K}_- such that

- (i) $\mathcal{K} = \mathcal{K}_+[\oplus]\mathcal{K}_-$
- (ii) $(\mathcal{K}_+, [., .])$ and $(\mathcal{K}_-, \langle[., .])$ are Hilbert spaces

- (2) For a Krein space let P_+ and P_- be the orthogonal projections onto \mathcal{K}_+ and \mathcal{K}_- respectively.

- (3) If $\min[\dim(\mathcal{K}_+), \dim(\mathcal{K}_-)] = k < \infty$, then \mathcal{K} is called a **Pontrjagin space**.

Remark A.51

For x an element of a Krein space exist unique elements $x_+ \in \mathcal{K}_+$ and $x_- \in \mathcal{K}_-$ such that $x = x_+ + x_-$. Then with the inner product

$$\langle x, y \rangle := [x_+, y_+] \langle [x_-, y_-] \tag{A.12}$$

the space $(\mathcal{K}, \langle[., .])$ is a Hilbert space. If two vectors x and y are orthogonal w.r.t. $\langle[., .]$ we write $x \perp y$ as usual.

Definition A.52

The **canonical symmetry** is the mapping $J := P_+ \langle [P_-$.

For the inner products $\langle[., .]$ and $[., .]$ we have with (A.12)

$$\langle x, y \rangle = [P_+x, y] \langle [P_-x, y] = [Jx, y]$$

and also $[x, y] = \langle Jx, y \rangle$.

ADJOINT OPERATORS ON KREIN SPACES

Definition A.53

- (1) For a linear operator A on a Krein space \mathcal{K} set

$$\mathcal{D}^+ := \{y \in \mathcal{K} | \forall x \in \mathcal{D}(A) : \exists z \in \mathcal{K} : [Ax, y] = [x, z]\}$$

and define the **adjoint operator** A^+ of A by $A^+y = z$ with domain \mathcal{D}^+ .

- (2) An operator A is called **self-adjoint** if $A = A^+$.

Theorem A.54

The spectrum of a self-adjoint operator A on a Pontrjagin space Π_k , $k \in \mathbb{N}$, is symmetric w.r.t. the real axis; every non-real number is either an eigenvalue or a regular point of A , in the UHP exist at most k eigenvalues.

And for the eigenvectors we have

Proposition A.55

Let $A = A^+$.

- (1) The eigenvectors u and v to eigenvalues λ and μ with $\lambda \neq \bar{\mu}$ are $[\cdot, \cdot]$ -orthogonal.
- (2) The eigenvectors to complex eigenvalues have zero $[\cdot, \cdot]$ -norm.

A.6 Hardy spaces

The definitions and facts listed in this Section are from Helson[12] and Hoffman[15]. Let $\mathcal{L}_2(d\lambda)$ be the space \mathcal{L}_2 on the unit circle $[\Leftarrow\pi, \pi]$ with measure $\frac{1}{2\pi}d\theta$.

Definition A.56

- (1) The **Hardy space** H_+^2 is the space of all complex-valued functions which are analytic *inside* the unit circle $|z| = 1$ and for which the limit

$$\|f\|_{H_+^2}^2 := \lim_{r \rightarrow 1^-} \|f_r\|_2^2 = \lim_{r \rightarrow 1^-} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta$$

exists; this limit defines a norm on the Hilbert space H_+^2 .

- (2) The **Hardy space** H_-^2 is the space of all complex-valued functions which are analytic *outside* the unit circle $|z| = 1$ and for which the limit

$$\|f\|_{H_-^2} := \lim_{r \rightarrow 1^+} \|f_r\|_2^2 = \lim_{r \rightarrow 1^+} \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(re^{i\theta})|^2 d\theta$$

exists; again, $(H_-^2, \|\cdot\|_{H_-^2})$ is a Hilbert space.

For almost every $\theta \in [\Leftarrow\pi, \pi]$ the radial limits

$$\begin{aligned} \tilde{f}(\theta) &= \lim_{r \rightarrow 1^-} f(re^{i\theta}) & f \in H_+^2 \\ \tilde{f}(\theta) &= \lim_{r \rightarrow 1^+} f(re^{i\theta}) & f \in H_-^2 \end{aligned}$$

exist and define \mathcal{L}_2 functions on the circle, and in fact

$$\|\tilde{f}\|_2 = \|f\|_{H_{\pm}^2}$$

The Hardy spaces H_+^2 and H_-^2 are often associated with the boundary functions in \mathcal{L}_2 of the circle rather than the analytic functions inside and outside the unit circle (see e.g. Helson[12]).

Definition A.57

(1) The **shift operator** χ on $\mathcal{L}_2(d\lambda)$ is defined by

$$(\chi f)(e^{i\theta}) = e^{i\theta} f(e^{i\theta})$$

(2) A subspace $M \subset \mathcal{L}_2(d\lambda)$ is said to be **simply invariant** if $\chi M \subset M$.

(3) An analytic function q inside the unit circle with $|q(z)| < 1$, $|z| < 1$, and $|q(e^{i\theta})| = 1$ almost everywhere on the unit circle is called an **inner function**.

A relation between inner functions and simply invariant spaces is given in the following theorem (Helson[12])

Theorem A.58

The simply invariant subspaces of $\mathcal{L}_2(d\lambda)$ are precisely the subspaces of the form qH_+^2 , where q is an inner function determined uniquely up to a constant of modulus 1.

Definition A.59

A **Blaschke product** is an infinite product of the form

$$B(z) = z^k \prod_{j=1}^{\infty} \left(\frac{a_j \Leftrightarrow z \bar{a}_j}{1 \Leftrightarrow \bar{a}_j z |a_j|} \right)^{p_j}$$

where $k \in \mathbb{N}_0$, $p_j \in \mathbb{N}$, and the zeros a_j of $B(z)$ satisfy

$$0 < |a_j| < 1 \quad \wedge \quad \sum_{j=1}^{\infty} (1 \Leftrightarrow |a_j|^{p_j}) < \infty$$

The inner functions are then characterised by

Theorem A.60

An inner function q is uniquely expressible in the form $q(z) = B(z)S(z)$, where $B(z)$ is a Blaschke product and $S(z)$ is an inner function without zeroes and positive (thus real) at the origin, also called a singular function.

For the proof of above factorisation theorem and the following definition and theorem see Hoffman[15].

Definition A.61

The Hardy space H_+^2 (H_-^2) in the UHP (LHP) is the space of all complex-valued functions analytic in the UHP (LHP) for which the \mathcal{L}_2 -norms

$$\int_{-\infty}^{\infty} |f(x + iy)|^2 dx$$

are bounded for $y > 0$ ($y < 0$).

It holds the important Paley-Wiener theorem

Theorem A.62 (Paley-Wiener)

A complex-valued function f analytic in the UHP is in H_+^2 if and only if f is of the form

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \hat{f}(t) e^{izt} dt$$

with $\text{Im}(z) > 0$ and some function $\hat{f} \in \mathcal{L}_2[0, \infty)$.

Appendix B

Attachments to Chapter 3

The graphs listed below are produced with help of the mathematical software 'Matlab', here we give a pseudo-code of the programme.

```
input  $r, \omega_2, \varkappa_0$ , number of steps  $n$ , step-size  $\Delta \varkappa$ 
for  $k = 1, \dots, n$ 
  construct  $\Omega = \mathcal{A}(\varkappa_0 + (k \Leftrightarrow 1)\Delta \varkappa)$ 
  find eigenvalues of  $\Omega$ 
  plot eigenvalues ('o' for non-negative imaginary part,
    '* ' for negative imaginary part)
end loop
```

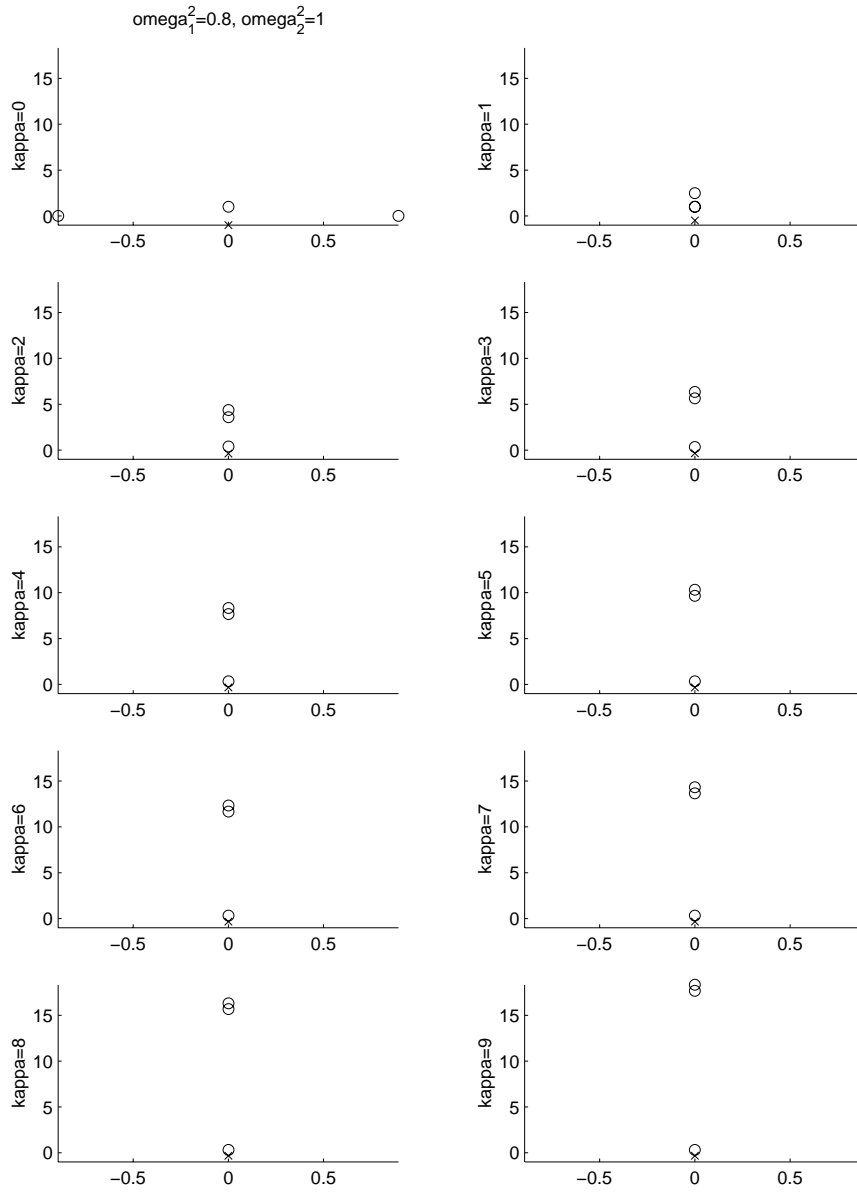


Figure B.1: $\omega_1^2 = 0.8$ and $\omega_2^2 = 1$, $\varkappa = 0, \dots, 9$

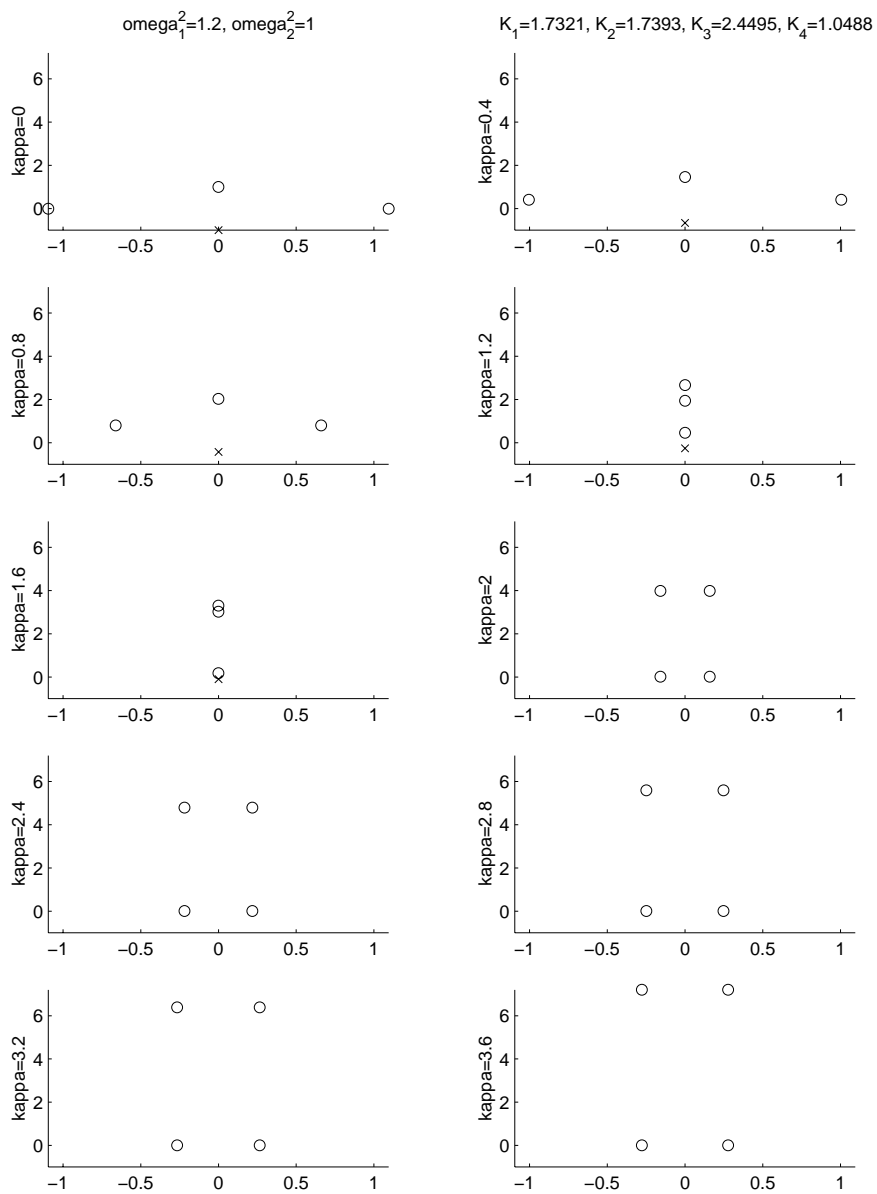


Figure B.2: $\omega_1^2 = 1.2$ and $\omega_2^2 = 1$, $\varkappa = 0, \dots, 3.6$

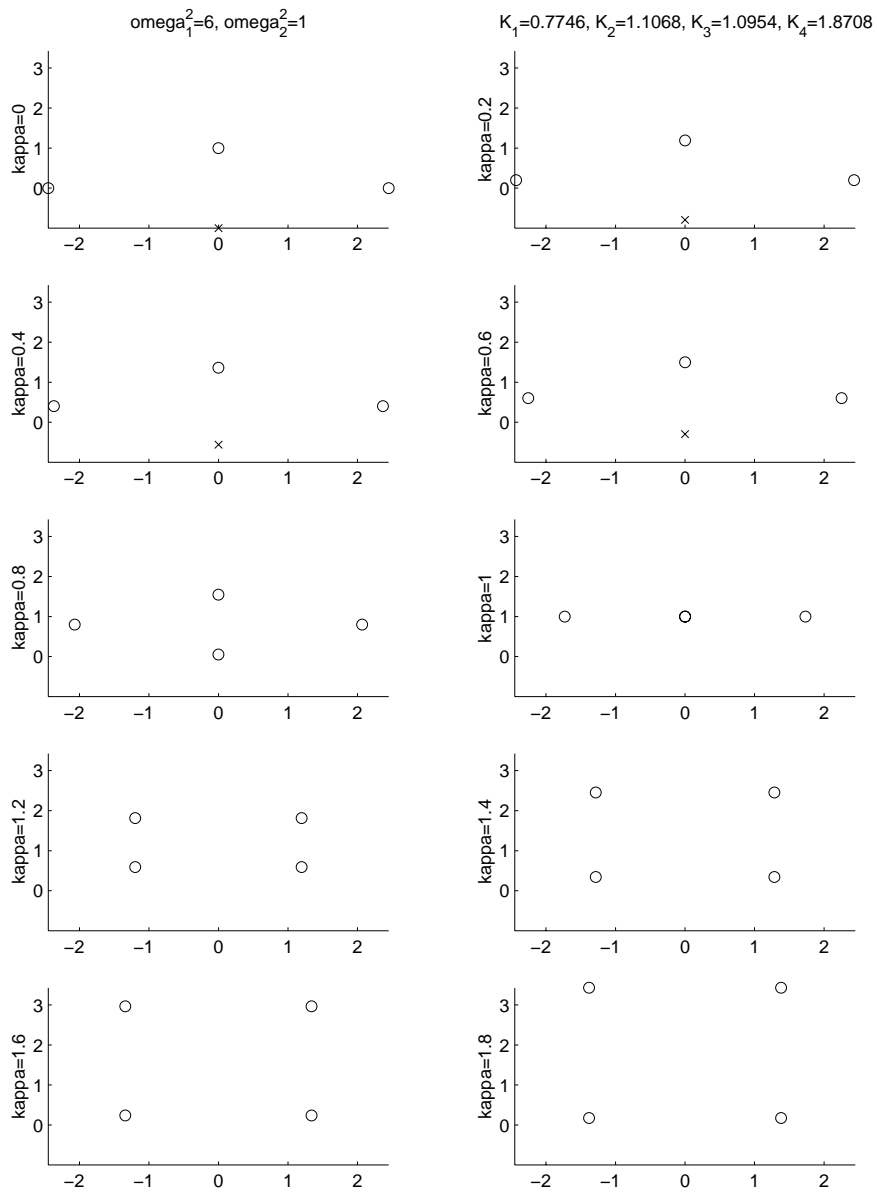


Figure B.3: $\omega_1^2 = 6$ and $\omega_2^2 = 1$, $\varkappa = 0, \dots, 1.8$

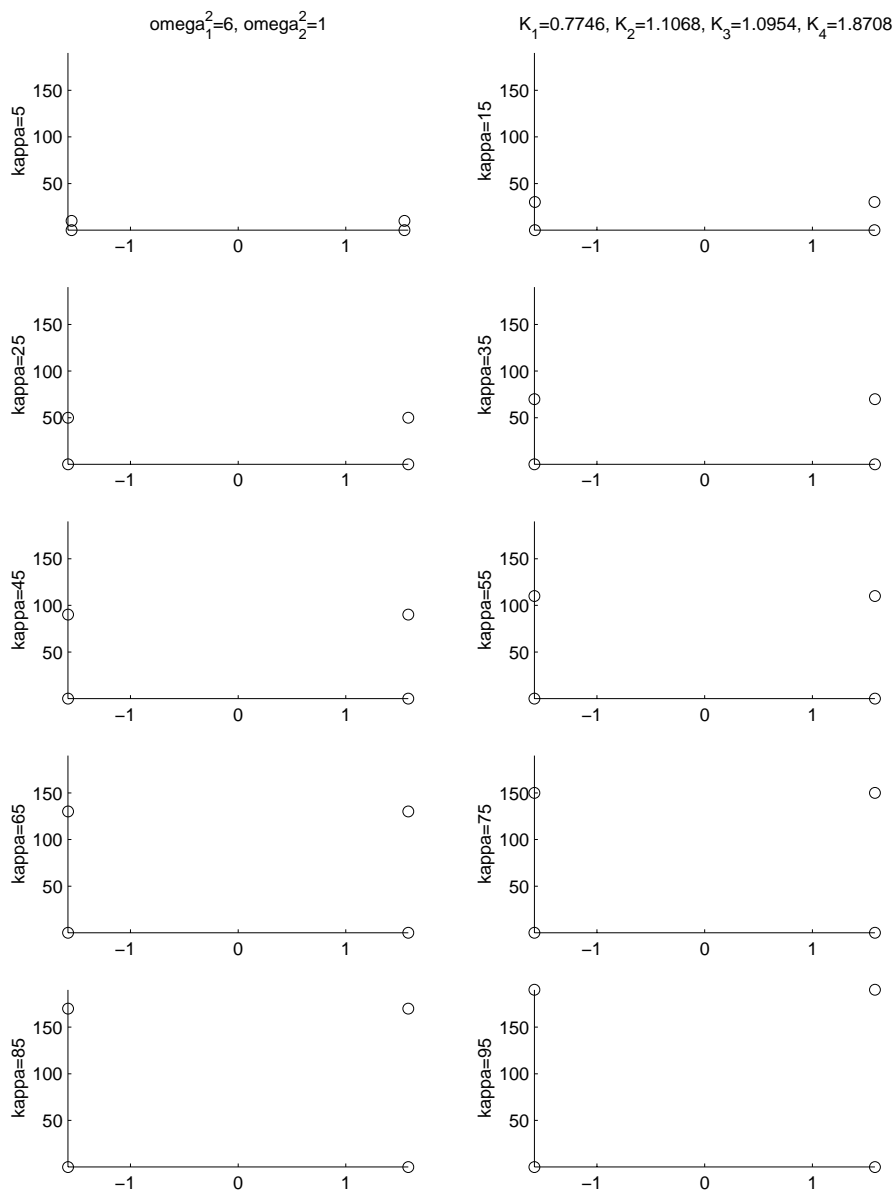


Figure B.4: $\omega_1^2 = 6$ and $\omega_2^2 = 1$, $\varkappa = 5, \dots, 95$

Appendix C

Attachments to Chapter 4

C.1 General calculations

C.1.1 The closure of the domain of A

The domain of A is (cf. Section 4.1)

$$\mathcal{D}(A) = \left\{ \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} : u_0 \in W_2^2, u_1 \in W_2^1, (u_0)_x \Leftrightarrow h u_0|_0 = 0, u_0|_N = 0, u_1|_N = 0 \right\},$$

the range is

$$\mathcal{C} := \left\{ \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} : u_0 \in W_2^1, u_1 \in \mathcal{L}_2, u_0|_N = 0 \right\},$$

and the energy form $[\cdot, \cdot]$ on \mathcal{C} is defined by

$$\left[\begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \right] = \frac{1}{2} \left(h|u_0|^2|_0 + \int_0^N |(u_0)_x|^2 + \varrho(x)|u_1|^2 dx \right)$$

Claim

The closure of $\mathcal{D}(A)$ w.r.t. the energy form is \mathcal{C} .

Sketch of the proof:

The \mathcal{L}_2 -closure of the second component of $\mathcal{D}(A)$, $\{z \in W_2^1 : z|_N = 0\}$ is \mathcal{L}_2 . So we need to show that the closure of $X = \{z \in W_2^2 : z' \Leftrightarrow h z|_0 = 0, z|_N = 0\}$ w.r.t. the norm

$$\|z\|^2 = \frac{1}{2} h|z|^2|_0 + \int_0^N |z'|^2 dx$$

is the space $Y = \{y \in W_2^1 : y|_N = 0\}$. Note that the norm $\|\cdot\|$ is not the actual W_2^1 norm (see Appendix A.4). Now let $y \in Y$ and $\varepsilon > 0$.

Since W_2^2 is dense in W_2^1 w.r.t. the W_2^1 -norm $\|\cdot\|_{W_2^1}$ (cf. Theorem A.38), we can find for $\delta > 0$ a function $z_1 \in W_2^1[\delta, N]$ such that $z_1|_N = 0$ and

$$\int_\delta^N |y' \Leftrightarrow z_1'|^2 dx \leq \|y \Leftrightarrow z_1\|_{W_2^1}^2 < \frac{\varepsilon}{2}$$

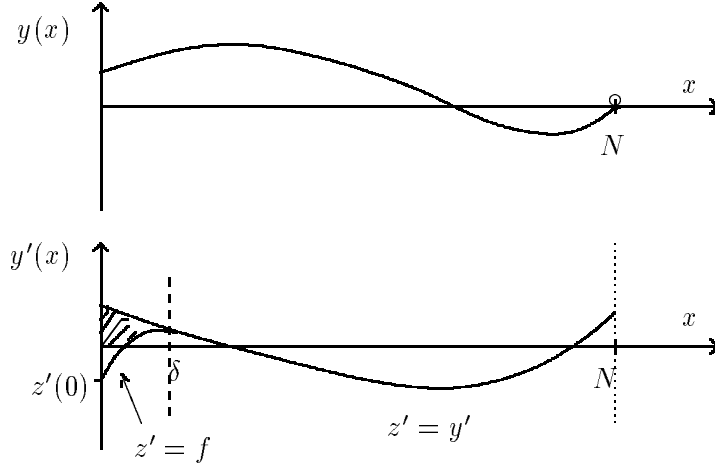


Figure C.1: The graphs of y and z for $h < 0$.

Define $z_2 \in X$ with $z_2(0) = y(0)$ and $z_2|_{[\delta, N]} = z_1$, then have

$$\|z_2 \Leftrightarrow y\|^2 \leq \int_0^\delta |z_2' \Leftrightarrow y'|^2 dx + \frac{\varepsilon}{2}$$

Because $z_1 \in X$ have $z_2'|_0 = h z_2(0) = h y(0)$.

With a construction similar to the one with which one shows that the set $\{f \in \mathcal{L}_2 : f(0) = 0\}$ is dense in \mathcal{L}_2 we can find a function with $f \in W_2^1$ defined on the interval $[0, \delta]$ and with $f(0) = h z_2(0)$ such that

$$\int_0^\delta |f \Leftrightarrow y'|^2 dx < \frac{\varepsilon}{2},$$

and so that the function

$$z(x) = \begin{cases} y(0) & x = 0 \\ \int_0^x f(t) dt & 0 \leq x \leq \delta \\ z_1(x) & \delta \leq x \leq N \end{cases}$$

is in W_2^2 , Fig. C.1 illustrates this construction. Then actually $z \in X$ and $\|z \Leftrightarrow y\|^2 < \varepsilon$, which is the required result.

C.1.2 The energy norms of the eigenvectors U_s, V_s and Ψ_\pm

(I) For the $\|\cdot\|_1$ -norms of the eigenvectors U_s of A_1 we have from (4.14)

$$\begin{aligned} \|U_s\|_1^2 &= \|U_{-s}\|_1^2 = \langle u_s, u_s \rangle_{\rho^2} = \\ &= \rho^2 \int_0^N \cos^2(\lambda_s \rho x) + \frac{2h_1}{\lambda_s \rho} \sin(\lambda_s \rho x) \cos(\lambda_s \rho x) + \frac{h_1^2}{\lambda_s^2 \rho^2} \sin^2(\lambda_s \rho x) dx \\ &= \rho^2 \left[\frac{N}{2} + \frac{1}{4\lambda_s \rho} \sin(2\lambda_s \rho N) + \frac{h_1}{\lambda_s^2 \rho^2} \sin^2(\lambda_s \rho N) \right. \\ &\quad \left. + \frac{h_1^2}{\lambda_s^2 \rho^2} \left(\frac{N}{2} \Leftrightarrow \frac{1}{4\lambda_s \rho} \sin(2\lambda_s \rho N) \right) \right]. \end{aligned}$$

With (cf. equation (4.29))

$$\sin(2\lambda_s \rho N) = \frac{2 \tan(\lambda_s \rho N)}{1 + \tan^2(\lambda_s \rho N)} = \frac{\Leftrightarrow 2\lambda_s \rho h_1}{h_1^2 + \lambda_s^2 \rho^2}$$

and

$$\frac{h_1}{\lambda_s \rho} \sin^2(\lambda_s \rho N) = \Leftrightarrow \frac{1}{2} \sin(2\lambda_s \rho N) = \frac{\lambda_s \rho h_1}{h_1^2 + \lambda_s^2 \rho^2}$$

we get

$$\|U_{\pm s}\|_1^2 = \rho^2 \left[\frac{N(\lambda_s^2 \rho^2 + h_1^2)}{2\lambda_s^2 \rho^2} + \frac{h_1}{2(\lambda_s^2 \rho^2 + h_1^2)} \left(1 + \frac{h_1^2}{\lambda_s^2 \rho^2}\right) \right]$$

and thus

$$\|U_s\|_1^2 = \|U_{-s}\|_1^2 = \frac{\rho^2 N}{2} + h_1 \frac{N h_1 + 1}{2\lambda_s^2} \geq \frac{\rho^2 N}{2}.$$

(II) Similar calculations yield for the eigenvectors V_s of A_2

$$\|V_s\|_2^2 = \|V_{-s}\|_2^2 = \frac{\rho^2 N}{2} + h_2 \frac{N h_2 + 1}{2\tau_s^2} \geq \frac{\rho^2 N}{2}.$$

(III) And for the $\|\cdot\|_J$ -norms of Ψ_+ and Ψ_- we get

$$\begin{aligned} \|\Psi_+\|_J^2 &= \|\Psi_-\|_J^2 = [J\Psi_+, \Psi_+]_2 \\ &= \frac{1}{2} \left(\frac{1}{\mu^2} \langle \Leftrightarrow L_2 e_\mu, e_\mu \rangle_{\rho^2} + \langle e_\mu, e_\mu \rangle_{\rho^2} \right) = \langle e_\mu, e_\mu \rangle_{\rho^2} \\ &= \rho^2 \int_0^N \cosh^2(\mu \rho x) + \frac{2h_2}{\mu \rho} \sinh(\mu \rho x) \cosh(\mu \rho x) + \frac{h_2^2}{\mu^2 \rho^2} \sinh^2(\mu \rho x) dx \\ &= \rho^2 \left[\left(\frac{\sinh(2\mu \rho N)}{4\mu \rho} + \frac{N}{2} \right) + \frac{h_2 \sinh^2(\mu \rho N)}{\mu^2 \rho^2} + \right. \\ &\quad \left. + \frac{h_2^2}{\mu^2 \rho^2} \left(\frac{\sinh(2\mu \rho N)}{4\mu \rho} \Leftrightarrow \frac{N}{2} \right) \right] \end{aligned}$$

With (cf. equation (4.33))

$$\sinh^2(\mu \rho N) = \frac{\tanh^2(\mu \rho N)}{1 \Leftrightarrow \tanh^2(\mu \rho N)} = \frac{\mu^2 \rho^2}{h_2^2 \Leftrightarrow \mu^2 \rho^2}$$

and

$$\frac{\sinh(2\mu \rho N)}{4\mu \rho} = \Leftrightarrow \frac{\cosh^2(\mu \rho N)}{2h_2} = \frac{\Leftrightarrow h_2}{2(h_2^2 \Leftrightarrow \mu^2 \rho^2)}$$

we get

$$\|\Psi_{\pm}\|_J^2 = \rho^2 \left[\frac{N}{2} \left(1 \Leftrightarrow \frac{h_2^2}{\mu^2 \rho^2} \right) \Leftrightarrow \frac{h_2}{2\mu^2 \rho^2} \right] = \frac{\rho^2 N}{2} \Leftrightarrow h_2 \frac{N h_2 + 1}{2\mu^2}.$$

C.1.3 The restriction of A_2 onto Π_-

Choose

$$\mathcal{B} := \left\{ \begin{pmatrix} e_\mu \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e_\mu \end{pmatrix} \right\}$$

as the basis for Π_- , then the restriction of A_2 onto the space Π_- is represented by the matrix

$$A_2|_{\Pi_-} = i \begin{pmatrix} 0 & \Leftrightarrow 1 \\ \Leftrightarrow \mu^2 & 0 \end{pmatrix}$$

For $a, b \in \mathbb{C}$ define

$$\begin{pmatrix} a \\ b \end{pmatrix} := \begin{pmatrix} ae_\mu \\ be_\mu \end{pmatrix}.$$

$A_2|_{\Pi_-}$ is anti-hermitian w.r.t. $[\cdot, \cdot]_J$, since for $a, b, c, d \in \mathbb{C}$ we have

$$\begin{aligned} \left[\begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right]_J &= \left[\begin{pmatrix} \Leftrightarrow a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right]_2 \\ &= \frac{1}{2} (\langle \Leftrightarrow a L e_\mu, c e_\mu \rangle_{\rho^2} + \langle b e_\mu, d e_\mu \rangle_{\rho^2}) \\ &= \frac{1}{2} (\mu^2 a \bar{c} + b \bar{d}) \|e_\mu\|_{\mathcal{L}_2}^2, \end{aligned}$$

and then

$$\begin{aligned} \left[A_2|_{\Pi_-} \begin{pmatrix} a \\ b \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right]_J &= \left[\begin{pmatrix} \Leftrightarrow i b \\ \Leftrightarrow \mu^2 a \end{pmatrix}, \begin{pmatrix} c \\ d \end{pmatrix} \right]_J \\ &= \frac{1}{2} (\mu^2 \Leftrightarrow i b \bar{c} \Leftrightarrow i \mu^2 a \bar{d}) \|e_\mu\|_{\mathcal{L}_2}^2 \\ &= \frac{1}{2} (b \overline{i \mu^2 c} + \mu^2 a \overline{id}) \|e_\mu\|_{\mathcal{L}_2}^2 \\ &= \left[\begin{pmatrix} a \\ b \end{pmatrix}, \Leftrightarrow \begin{pmatrix} id \\ i \mu^2 c \end{pmatrix} \right]_J = \Leftrightarrow \left[\begin{pmatrix} a \\ b \end{pmatrix}, A_2|_{\Pi_-} \begin{pmatrix} c \\ d \end{pmatrix} \right]_J \end{aligned}$$

Thus

$$(A_2|_{\Pi_-})^* = (\overline{A_2|_{\Pi_-}})^t = \Leftrightarrow A_2|_{\Pi_-}$$

C.1.4 The adjoint operators of A_0

Here we will verify that

$$U \in \mathcal{D}_0, V \in \mathcal{D}(A_1) \Rightarrow [A_0 U, V]_1 = [U, A_1 V]_1$$

$$\text{and } U \in \mathcal{D}_0, V \in \mathcal{D}(A_2) \Rightarrow [A_0 U, V]_2 = [U, A_2 V]_2$$

or, in other words,

$$A_1 \subset A_0^* \quad \wedge \quad A_2 \subset A_0^+.$$

We have

$$\begin{aligned} \mathcal{D}_0 &= \left\{ \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} : u_0 \in W_2^2, u_1 \in W_2^1, (u_0)_x|_0 = u_0|_0 = 0, u_0|_N = 0, u_1|_N = 0 \right\} \\ \mathcal{D}(A_1) &= \left\{ \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} : v_0 \in W_2^2, v_1 \in W_2^1, (v_0)_x \Leftrightarrow h_1 v_0|_0 = 0, u_0|_N = 0, v_1|_N = 0 \right\}. \end{aligned}$$

and we let $U = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} \in \mathcal{D}_0$, $V = \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \in \mathcal{D}(A_1)$. Then we get for the value of $[A_0 U, V]_1$

$$\begin{aligned} 2[A_0 U, V]_1 &= 2 \left[\begin{pmatrix} 0 & \Leftrightarrow i \\ iL & 0 \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \end{pmatrix}, \begin{pmatrix} v_0 \\ v_1 \end{pmatrix} \right]_1 \\ &= h_1 (\Leftrightarrow i u_1) \overline{v_0}|_0 + \int_0^N (\Leftrightarrow i u_1)_x \overline{(v_0)_x} + \rho^2 (i L_1 u_0) \overline{v_1} dx \\ &= h_1 (\Leftrightarrow i u_1) \overline{v_0}|_0 + \left(h_1 (i u_1) \overline{v_0}|_0 \int_0^N \rho^2 u_1 \overline{(i L_1 v_0)} dx \right) + \int_0^N (u_0)_x \overline{(\Leftrightarrow i v_1)_x} dx. \end{aligned}$$

Since $u_0|_N = 0$ we have

$$\begin{aligned} 2[A_0 U, V]_1 &= h_1 u_0 \overline{(\Leftrightarrow i v_1)}|_0 + \int_0^N (u_0)_x \overline{(\Leftrightarrow i v_1)_x} + \rho^2 u_1 \overline{(i L_1 v_0)} dx \\ &= 2[U, A_1 V]_1 \end{aligned}$$

Thus we have the required result

$$U \in \mathcal{D}_0, V \in \mathcal{D}(A_1) \Rightarrow [A_0 U, V]_1 = [U, A_1 V]_1$$

i.e. $A_0 \subset A_1^*$ in $(\mathcal{C}, [., .]_1)$. Similar calculations yield $A_0 \subset A_2^+$ in $(\mathcal{C}, [., .]_2)$.

C.1.5 Estimation of $\|U_{s+1} - V_s\|_j^2, j = 1, 2$

Here we will verify that

$$\lim_{s \rightarrow \infty} \|U_{s+1} \Leftrightarrow V_s\|_j^2 = 0, j = 1, 2$$

The similar will then be true for $s < 0$, i.e.

$$\lim_{s \rightarrow -\infty} \|U_{s-1} \Leftrightarrow V_s\|_j^2 = 0, j = 1, 2$$

Let $s > 0$. For the $[., .]_1$ -norm of the difference of the eigenvectors

$$U_{s+1} = \begin{pmatrix} \frac{1}{i\lambda_{s+1}} u_{s+1} \\ u_{s+1} \end{pmatrix} \quad \text{with} \quad u_{s+1}(x) = \cos(\lambda_{s+1} \rho x) + \frac{h_1}{\lambda_{s+1} \rho} \sin(\lambda_{s+1} \rho x)$$

of A_1 and

$$V_s = \begin{pmatrix} \frac{1}{i\tau_s} v_s \\ v_s \end{pmatrix} \quad \text{with} \quad v_s(x) = \cos(\tau_s \rho x) + \frac{h_2}{\tau_s \rho} \sin(\tau_s \rho x)$$

of A_2 we have (cf. (4.6))

$$\begin{aligned} \|U_{s+1} \Leftrightarrow V_s\|_1^2 &= \frac{h_1}{2} \left| \frac{1}{\lambda_{s+1}} \Leftrightarrow \frac{1}{\tau_s} \right|^2 + \\ &+ \frac{1}{2} \int_0^N \left| \frac{u'(x)}{\lambda_{s+1}} \Leftrightarrow \frac{v'_s(x)}{\tau_s} \right|^2 + \rho^2 |u(x) \Leftrightarrow v_s(x)|^2 dx \end{aligned} \quad (\text{C.1})$$

For the first term in (C.1) we have

$$\left| \frac{1}{\lambda_{s+1}} \Leftrightarrow \frac{1}{\tau_s} \right| = \frac{|\lambda_{s+1} \Leftrightarrow \tau_s|}{|\lambda_{s+1}\tau_s|} \quad (\text{C.2})$$

We will give further estimations later. In order to estimate the second term

$$\int_0^N \left| \frac{u'_{s+1}(x)}{\lambda_{s+1}} \Leftrightarrow \frac{v'_s(x)}{\tau_s} \right|^2 dx$$

in (C.1) we need to calculate

$$\begin{aligned} \left| \frac{u'_{s+1}(x)}{\lambda_{s+1}} \Leftrightarrow \frac{v'_s(x)}{\tau_s} \right| &= \left| \left[\Leftrightarrow \rho \sin(\lambda_{s+1}\rho x) + \frac{h_1}{\lambda_{s+1}} \cos(\lambda_{s+1}\rho x) \right] + \right. \\ &\quad \left. \Leftrightarrow \left[\Leftrightarrow \rho \sin(\tau_s\rho x) + \frac{h_2}{\tau_s} \cos(\tau_s\rho x) \right] \right| \\ &\leq 2\rho \left| \cos\left(\frac{\tau_s + \lambda_{s+1}}{2}\rho x\right) \sin\left(\frac{\tau_s \Leftrightarrow \lambda_{s+1}}{2}\rho x\right) \right| + \frac{h_1}{\lambda_{s+1}} + \frac{|h_2|}{\tau_s} \\ &\leq \rho^2 N |\tau_s \Leftrightarrow \lambda_{s+1}| + \frac{h_1 \Leftrightarrow h_2}{\tau_s} \end{aligned} \quad (\text{C.3})$$

Inequality (C.2) is true since $\lambda_{s+1} > \tau_s$, $|\sin(\xi)| \leq |\xi|$ and $0 \leq x \leq N$.

In order to estimate the third term

$$\int_0^N \rho^2 |u_{s+1}(x) \Leftrightarrow v_s(x)|^2 dx$$

in (C.1) we observe that

$$\begin{aligned} \rho |u_{s+1}(x) \Leftrightarrow v_s(x)| &= \rho \left| \cos(\lambda_{s+1}\rho x) + \frac{h_1 \sin(\lambda_{s+1}\rho x)}{\lambda_{s+1}\rho} \Leftrightarrow \cos(\tau_s\rho x) \Leftrightarrow \frac{h_2 \sin(\tau_s\rho x)}{\tau_s\rho} \right| \\ &\leq \rho \left| 2 \sin\left(\frac{\lambda_{s+1} + \tau_s}{2}\rho x\right) \sin\left(\frac{\lambda_{s+1} \Leftrightarrow \tau_s}{2}\rho x\right) \right| + \frac{h_1}{\lambda_{s+1}} + \frac{|h_2|}{\tau_s} \\ &\leq \rho^2 N |\lambda_{s+1} \Leftrightarrow \tau_s| + \frac{h_1 \Leftrightarrow h_2}{\tau_s} \end{aligned} \quad (\text{C.4})$$

For $|\lambda_{s+1} \Leftrightarrow \tau_s|$ have with (4.24) and (4.31)

$$|\lambda_{s+1} \Leftrightarrow \tau_s| = |\varepsilon_{s+1} + \delta_s| \leq \frac{(h_1 \Leftrightarrow h_2)N}{s\pi} \quad (\text{C.5})$$

and recall (4.25) and (4.32)

$$|\lambda_{s+1}| \geq \frac{s\pi}{N\rho} \quad \wedge \quad |\tau_s| \geq \frac{s\pi}{N\rho} \quad (\text{C.6})$$

Substitute (C.2) - (C.6) in (C.1)

$$\begin{aligned} \|U_{s+1} \Leftrightarrow V_s\|_1^2 &\leq \frac{h_1 |\lambda_{s+1} \Leftrightarrow \tau_s|^2}{2 |\lambda_{s+1}\tau_s|^2} + N \left(\rho^2 N |\lambda_{s+1} \Leftrightarrow \tau_s| + \frac{h_1 \Leftrightarrow h_2}{|\tau_s|} \right)^2 \\ &\leq \frac{h_1 (h_1 \Leftrightarrow h_2)^2 N^2 N^4 \rho^4}{2 s^2 \pi^2 s^4 \pi^4} + N \left(\rho^2 N \frac{(h_1 \Leftrightarrow h_2)N}{s\pi} + \frac{(h_1 \Leftrightarrow h_2)N\rho}{s\pi} \right)^2 \\ &\leq \frac{h_1 (h_1 \Leftrightarrow h_2)^2 N^6 \rho^4}{2s^6 \pi^6} + \frac{(h_1 \Leftrightarrow h_2)^2}{s^2 \pi^2} N^3 \rho^2 (N\rho + 1)^2 \end{aligned} \quad (\text{C.7})$$

For the $[\cdot, \cdot]_2$ -norm have clearly

$$\|U_{s+1} \Leftrightarrow V_s\|_2^2 \leq \frac{h_2(h_1 \Leftrightarrow h_2)^2 N^6 \rho^4}{2s^6 \pi^6} + \frac{(h_1 \Leftrightarrow h_2)^2}{s^2 \pi^2} N^3 \rho^2 (N\rho + 1)^2$$

So that we get the desired result

$$\lim_{s \rightarrow \infty} \|U_{s+1} \Leftrightarrow V_s\|_j^2 = 0, \quad j = 1, 2,$$

in fact it is $\|U_{s+1} \Leftrightarrow V_s\|_j^2 = o(s^{-1})$, $j = 1, 2$.

C.2 The Galerkin method

C.2.1 The system matrix Γ

The approximative eigenvalues $\alpha_K^{(j)}$ are the eigenvalues of the $(4K+4) \times (4K+4)$ -matrix

$$\Gamma = \begin{pmatrix} \gamma_1 & \gamma_2 & 0 \\ \gamma_3 & \gamma_4 & \gamma_5 \\ 0 & \gamma_6 & \gamma_7 \end{pmatrix}$$

With the matrices $\gamma_1, \dots, \gamma_7$ as follows

$$\begin{aligned} \gamma_1 &= \begin{pmatrix} i\mu + i\mathcal{X} & 0 \\ 0 & \Leftrightarrow i\mu + i\mathcal{X} \end{pmatrix} \\ \gamma_2 &= \Leftrightarrow i\mathcal{X} \begin{pmatrix} b_{-K-1}^+ & \cdots & b_{-1}^+ & b_1^+ & \cdots & b_{K+1}^+ \\ b_{-K-1}^- & \cdots & b_{-1}^- & b_1^- & \cdots & b_{K+1}^- \end{pmatrix} \\ \gamma_3 &= \Leftrightarrow i\mathcal{X} \begin{pmatrix} \psi_{+}^{-K-1} & \cdots & \psi_{+}^{-1} & \psi_{+}^1 & \cdots & \psi_{+}^{K+1} \\ \psi_{-}^{-K-1} & \cdots & \psi_{-}^{-1} & \psi_{-}^1 & \cdots & \psi_{-}^{K+1} \end{pmatrix}^t \\ \gamma_4 &= \text{diag}(\lambda_{-K-1} + i\mathcal{X}, \dots, \lambda_{-1} + i\mathcal{X}, \lambda_1 + i\mathcal{X}, \dots, \lambda_{K+1} + i\mathcal{X}) \\ \gamma_7 &= \text{diag}(\tau_{-K} + i\mathcal{X}, \dots, \tau_{-1} + i\mathcal{X}, \tau_1 + i\mathcal{X}, \dots, \tau_K + i\mathcal{X}) \\ \gamma_5 &= \Leftrightarrow i\mathcal{X} \begin{pmatrix} d_{-K}^{-K-1} & \cdots & d_{-1}^{-K-1} & d_1^{-K-1} & \cdots & d_K^{-K-1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ d_{-K}^{-1} & \cdots & d_{-1}^{-1} & d_1^{-1} & \cdots & d_K^{-1} \\ d_{-K}^1 & \cdots & d_{-1}^1 & d_1^1 & \cdots & d_K^1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ d_{-K}^{K+1} & \cdots & d_{-1}^{K+1} & d_1^{K+1} & \cdots & d_K^{K+1} \end{pmatrix} \\ \gamma_6 &= \Leftrightarrow i\mathcal{X} \begin{pmatrix} c_{-K-1}^{-K} & \cdots & c_{-1}^{-K} & c_1^{-K} & \cdots & c_{K+1}^{-K} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{-K-1}^{-1} & \cdots & c_{-1}^{-1} & c_1^{-1} & \cdots & c_{K+1}^{-1} \\ c_{-K-1}^1 & \cdots & c_{-1}^1 & c_1^1 & \cdots & c_{K+1}^1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ c_{-K-1}^K & \cdots & c_{-1}^K & c_1^K & \cdots & c_{K+1}^K \end{pmatrix} \end{aligned}$$

C.2.2 Estimation of Δ_K

The Galerkin method described and implemented in Section 4.4 produces approximate eigenvalues $\alpha_K^{(j)}$, $j = 1, \dots, 4K + 4$ for given choice of K . We will estimate the value

$$\Delta_K := \frac{\|\mathcal{A}\Phi_K \Leftrightarrow \alpha_K \Phi_K\|}{\|\Phi_K\|} \quad (\text{C.8})$$

with an uniform estimate, so that there is no need for the index j .

Recall from Section 4.4.2. that

$$\Phi_K = \begin{pmatrix} \sum_{s=-K-1, s \neq 0}^{K+1} E^s U_s \\ \sum_{t=-K, t \neq 0}^K F^t V_t + G^+ \Psi_+ + G^- \Psi_- \end{pmatrix}$$

and the components of $\mathcal{A}\Phi_K$ are

$$\begin{aligned} (\mathcal{A}\Phi_K)_1 &= \sum_{s=-K-1, s \neq 0}^{K+1} (\lambda_s + i\kappa \Leftrightarrow \alpha) E^s U_s \Leftrightarrow i\kappa \sum_{t=-K, t \neq 0}^K F^t V_t \Leftrightarrow i\kappa G^+ \Psi_+ \Leftrightarrow i\kappa G^- \Psi_- \\ (\mathcal{A}\Phi_K)_2 &= \Leftrightarrow i\kappa \sum_{s=-K-1, s \neq 0}^{K+1} E^s U_s + \sum_{t=-K, t \neq 0}^K (\tau_t + i\kappa \Leftrightarrow \alpha) F^t V_t + \\ &\quad + (i\mu + i\kappa \Leftrightarrow \alpha) G^+ \Psi_+ + (\Leftrightarrow i\mu + i\kappa \Leftrightarrow \alpha) G^- \Psi_- \end{aligned}$$

Due to orthogonality we have

$$\begin{aligned} \|\mathcal{A}\Phi_K \Leftrightarrow \alpha_K \Phi_K\|^2 &= \\ &= \sum_{s \in \mathbb{Z}^*} \left| \left\langle \mathcal{A}\Phi_K \Leftrightarrow \alpha_K \Phi_K, \begin{pmatrix} U_s \\ 0 \end{pmatrix} \right\rangle \right|^2 + \sum_{t \in \mathbb{Z}^*} \left| \left\langle \mathcal{A}\Phi_K \Leftrightarrow \alpha_K \Phi_K, \begin{pmatrix} 0 \\ V_t \end{pmatrix} \right\rangle \right|^2 + \\ &\quad + \left| \left\langle \mathcal{A}\Phi_K \Leftrightarrow \alpha_K \Phi_K, \begin{pmatrix} 0 \\ \Psi_+ \end{pmatrix} \right\rangle \right|^2 + \left| \left\langle \mathcal{A}\Phi_K \Leftrightarrow \alpha_K \Phi_K, \begin{pmatrix} 0 \\ \Psi_- \end{pmatrix} \right\rangle \right|^2, \end{aligned}$$

and with condition (4.78) we get

$$\begin{aligned} \|\mathcal{A}\Phi_K \Leftrightarrow \alpha_K \Phi_K\|^2 &= \\ &= \sum_{|s| \geq K+2} \left| \left\langle \mathcal{A}\Phi_K \Leftrightarrow \alpha_K \Phi_K, \begin{pmatrix} U_s \\ 0 \end{pmatrix} \right\rangle \right|^2 + \sum_{|t| \geq K+1} \left| \left\langle \mathcal{A}\Phi_K \Leftrightarrow \alpha_K \Phi_K, \begin{pmatrix} 0 \\ V_t \end{pmatrix} \right\rangle \right|^2 \end{aligned}$$

We make use of the orthogonality and Cauchy-Schwarz inequality to get (let in all sums

$s \neq 0$ and $t \neq 0$)

$$\begin{aligned}
& \|\mathcal{A}\Phi_K \Leftrightarrow \alpha_K \Phi_K\|^2 = \\
& = \varkappa^2 \sum_{|s| \geq K+2} \left| \left(\sum_{|t| \leq K} F^t [V_t, U_s]_1 \right) + G^+ [\Psi_+, U_s]_1 + G^- [\Psi_-, U_s]_1 \right|^2 + \\
& + \varkappa^2 \sum_{|t| \geq K+1} \left| \sum_{|s| \leq K+1} E^s [U_s, V_t]_2 \right|^2 \\
& \leq \varkappa^2 \sum_{|s| \geq K+2} \left[\left(\sum_{|t| \leq K} |F^t|^2 \sum_{|t| \leq K} |[V_t, U_s]_1|^2 \right)^{\frac{1}{2}} + |G^+| |[\Psi_+, U_s]_1| + |G^-| |[\Psi_-, U_s]_1| \right]^2 + \\
& + \varkappa^2 \sum_{|t| \geq K+1} \left[\sum_{|s| \leq K+1} |E^s|^2 \sum_{|s| \leq K+1} |[U_s, V_t]_2|^2 \right]
\end{aligned}$$

Since $|a+b+c|^2 \leq 3(|a|^2 + |b|^2 + |c|^2)$, $a, b, c \in \mathbb{C}$ and with (4.48)-(4.53) in Section 4.2.4.

$$\begin{aligned}
& \|\mathcal{A}\Phi_K \Leftrightarrow \alpha_K \Phi_K\|^2 \leq \\
& \leq 3\varkappa^2 \sum_{|s| \geq K+2} \left[\sum_{|t| \leq K} |F^t|^2 \sum_{|t| \leq K} |d_t^s|^2 \|U_s\|_1^4 + |G^+|^2 |\psi_+^s|^2 \|U_s\|_1^4 + |G^-|^2 |\psi_-^s|^2 \|U_s\|_1^4 \right] + \\
& + \varkappa^2 \sum_{|t| \geq K+1} \left[\sum_{|s| \leq K+1} |E^s|^2 \sum_{|s| \leq K+1} |c_s^t|^2 \|V_t\|_2^2 \right]
\end{aligned}$$

Observe that

$$\begin{aligned}
& \|\Phi_K\|^2 = \sum_{|s| \leq K+1} |E^s|^2 \|U_s\|_1^2 + \sum_{|t| \leq K} |F^t|^2 \|V_t\|_2^2 + |G^+|^2 \|\Psi_+\|_J^2 + |G^-|^2 \|\Psi_-\|_J^2 \\
\Rightarrow & \begin{cases} \sum_{|s| \leq K+1, s \neq 0} |E^s|^2 & \leq \frac{1}{\|U_s\|_1^2} \|\Phi_K\|^2 & \leq \frac{2}{\rho^2 N} \|\Phi_K\|^2 \\ \sum_{|t| \leq K, t \neq 0} |F^t|^2 & \leq \frac{1}{\|V_t\|_2^2} \|\Phi_K\|^2 & \leq \frac{2}{\rho^2 N} \|\Phi_K\|^2 \\ |G^\pm|^2 & \leq \frac{1}{\|\Psi_\pm\|_J^2} \|\Phi_K\|^2 \end{cases} \quad (\text{C.9})
\end{aligned}$$

Then

$$\begin{aligned}
& \|\mathcal{A}\Phi_K \Leftrightarrow \alpha_K \Phi_K\|^2 \leq \\
& \leq 3\varkappa^2 \|\Phi_K\|^2 \sum_{|s| \geq K+2} \left(\frac{2}{\rho^2 N} \sum_{|t| \leq K} |d_t^s|^2 \|U_s\|_1^4 + \frac{|\psi_+|^2 \|U_s\|_1^4}{\|\Psi_+\|_J^2} + \frac{|\psi_-|^2 \|U_s\|_1^4}{\|\Psi_-\|_J^2} \right) + \\
& + \varkappa^2 \|\Phi_K\|^2 \sum_{|t| \geq K+1} \left(\sum_{|s| \leq K+1} \frac{2}{\rho^2 N} |c_s^t|^2 \|V_t\|_2^2 \right) \quad (\text{C.10})
\end{aligned}$$

(I) Estimation of $|\lambda_s \Leftrightarrow \tau_t|$

For $|\lambda_s \Leftrightarrow \tau_t|$ we get with (4.23) and (4.30)

(1) For $s > 0 \wedge t < 0$ have

$$|\lambda_s \Leftrightarrow \tau_t| = \lambda_{|s|} + \tau_{|t|} = \frac{1}{\rho N} [(|s| + |t|)\pi + \varepsilon_s \Leftrightarrow \delta_t] \geq \frac{(|s| + |t| \Leftrightarrow 1)\pi}{\rho N}$$

For the following let $s, t > 0$.

(2) $s \geq t + 2$, then

$$|\lambda_s \Leftrightarrow \tau_t| = \frac{1}{\rho N} [(s \Leftrightarrow t \Leftrightarrow 1)\pi + \varepsilon_s + \delta_t] \geq \frac{(s \Leftrightarrow t \Leftrightarrow 1)\pi}{\rho N}$$

(3) $t = s = K + 1$, then

$$|\lambda_{K+1} \Leftrightarrow \tau_{K+1}| = \tau_{K+1} \Leftrightarrow \lambda_{K+1} = \frac{\pi \Leftrightarrow \varepsilon_{K+1} \Leftrightarrow \delta_{K+1}}{\rho N}$$

for $K + 1$ large enough have certainly

$$\frac{\pi}{\rho N} \geq \frac{\pi \Leftrightarrow \varepsilon_{K+1} \Leftrightarrow \delta_{K+1}}{\rho N} = \eta_{K+1} \frac{\pi}{\rho N}$$

with $0 < \eta_{K+1} < 1$ and $\eta_K \leq \eta_{K+1}$ for all $K \in N$.

In order to estimate

$$\eta_{K+1} \frac{\pi}{\rho N} \geq \tilde{\eta} \frac{\pi}{\rho N}$$

and then obtain

$$\frac{\pi \Leftrightarrow \varepsilon_{K+1} \Leftrightarrow \delta_{K+1}}{\rho N} \geq \tilde{\eta} \frac{\pi}{\rho N},$$

we require that $\varepsilon_{K+1} + \delta_{K+1} \leq \pi(1 \Leftrightarrow \tilde{\eta})$. Because of (4.24) and (4.31) this is fulfilled when

$$\begin{aligned} \frac{N}{(K+1)\pi} (h_1 \Leftrightarrow h_2) \leq \pi(1 \Leftrightarrow \tilde{\eta}) &\Leftrightarrow K+1 \geq \frac{N(h_1 \Leftrightarrow h_2)}{\pi^2(1 \Leftrightarrow \tilde{\eta})} \\ &\Leftrightarrow \tilde{\eta} \leq 1 \Leftrightarrow \frac{N(h_1 \Leftrightarrow h_2)}{(K+1)\pi^2} \end{aligned} \quad (\text{C.11})$$

(4) $t = s + 1 = K + 1$, then

$$|\lambda_s \Leftrightarrow \tau_t| = |\tau_{K+1} \Leftrightarrow \lambda_K| = \frac{1}{\rho N} [2\pi \Leftrightarrow \varepsilon_K \Leftrightarrow \delta_{K+1}] \geq \frac{\pi}{\rho N}$$

(5) $t \geq s + 2$, then

$$|\lambda_s \Leftrightarrow \tau_t| = \frac{1}{\rho N} [(t \Leftrightarrow s + 1)\pi \Leftrightarrow \varepsilon_s \Leftrightarrow \delta_t] \geq \frac{(t \Leftrightarrow s)\pi}{\rho N}$$

(II) Estimation of the coefficients

We substitute above expressions, (4.60)-(4.65) and

$$|\lambda_s| \geq \frac{(2|s| \Leftrightarrow 1)\pi}{2\rho N} \quad \wedge \quad |\tau_t| \geq \frac{|s|\pi}{\rho N}$$

from (4.25), (4.32) into (4.48), (4.51), (4.52) and (4.53) to get for d_i^s and c_s^t with $\Theta := \frac{(h_1 - h_2)^2 N^4 \rho^4}{\pi^4}$ (divided into five cases)

(1) Let $s > 0 \wedge t < 0$, so that with $s + |t| \Leftrightarrow 1 \geq |t|$, $s + |t| \Leftrightarrow 1 \geq s$ and $2s \Leftrightarrow 1 \geq s$ it follows

$$|d_t^s|^2 \|U_s\|_1^4 = \left(\frac{h_1 \Leftrightarrow h_2}{2\lambda_s \tau_t} + \frac{h_1 \Leftrightarrow h_2}{2\lambda_s (\lambda_s \Leftrightarrow \tau_t)} \right)^2 \leq 4\Theta \frac{1}{(2s \Leftrightarrow 1)^2 t^2} \quad (\text{C.12})$$

$$\begin{aligned} |c_s^t|^2 \|V_t\|_2^4 &= \left(\frac{h_1 \Leftrightarrow h_2}{2\lambda_s \tau_t} + \frac{h_1 \Leftrightarrow h_2}{2\tau_t (\lambda_s \Leftrightarrow \tau_t)} \right)^2 \\ &\leq \frac{1}{4} \Theta \frac{1}{t^2} \left(\frac{4}{(2s \Leftrightarrow 1)^2} + \frac{4}{(2s \Leftrightarrow 1)^2} + \frac{1}{s^2} \right) \end{aligned} \quad (\text{C.13})$$

(2) Let $s \geq t + 2$ and $s \geq K + 2$, then with $s \Leftrightarrow t \Leftrightarrow 1 \geq K + 1 \Leftrightarrow t$, then

$$\begin{aligned} |d_t^s|^2 \|U_s\|_1^4 &\leq \frac{(h_1 \Leftrightarrow h_2)^2}{4} \left(\frac{2N^2 \rho^2}{\pi^2 (2s \Leftrightarrow 1) |t|} + \frac{2N^2 \rho^2}{\pi^2 (2s \Leftrightarrow 1) (s \Leftrightarrow t \Leftrightarrow 1)} \right) \\ &\leq \Theta \frac{1}{(2s \Leftrightarrow 1)^2} \left(\frac{1}{t^2} + \frac{1}{(K + 1 \Leftrightarrow t)t} + \frac{1}{(K + 1 \Leftrightarrow t)^2} \right) \end{aligned} \quad (\text{C.14})$$

(3) Let $t = s = K + 1$, then

$$|c_{K+1}^{K+1}|^2 \|V_{K+1}\|_2^4 = \frac{(h_1 \Leftrightarrow h_2)^2}{4|\lambda_{K+1}|^2 |\lambda_{K+1} \Leftrightarrow \tau_{K+1}|^2} \leq \frac{\Theta}{(2K + 1)^2} \frac{1}{\eta_{K+1}^2} \quad (\text{C.15})$$

(4) Let $t = s + 1 = K + 1$, then

$$|c_K^{K+1}|^2 \|V_{K+1}\|_2^4 \leq \frac{\Theta}{(2K \Leftrightarrow 1)^2} \quad (\text{C.16})$$

(5) Let $t \geq s + 2$, $s \leq K \Leftrightarrow 1$, then with $t \Leftrightarrow s \geq K \Leftrightarrow s$ and $2s \Leftrightarrow 1 \geq s$, then

$$\begin{aligned} |c_s^t|^2 \|V_t\|_2^4 &\leq \frac{1}{4} \Theta \left(\frac{4}{(2s \Leftrightarrow 1)^2 t^2} + \frac{4}{t^2 (2s \Leftrightarrow 1) (t \Leftrightarrow s)} + \frac{1}{t^2 (t \Leftrightarrow s)^2} \right) \\ &\leq \frac{1}{4} \Theta \frac{1}{t^2} \left(\frac{4}{(2s \Leftrightarrow 1)^2} + \frac{4}{(2s \Leftrightarrow 1) (K \Leftrightarrow s)} + \frac{1}{(K \Leftrightarrow s)^2} \right) \end{aligned} \quad (\text{C.17})$$

And for ψ_+^s and ψ_-^s we have

$$\begin{aligned} |\psi_+^s|^2 \|U_s\|_1^4 &= \left| \frac{h_1 \Leftrightarrow h_2}{2\mu(\lambda_s + i\mu)} \right|^2 \leq \frac{(h_1 \Leftrightarrow h_2)^2 \rho^2 N^2}{\mu^2 \pi^2 (2|s| \Leftrightarrow 1)^2} \\ |\psi_-^s|^2 \|U_s\|_1^4 &= \left| \frac{h_1 \Leftrightarrow h_2}{2\mu(\lambda_s \Leftrightarrow i\mu)} \right|^2 \leq \frac{(h_1 \Leftrightarrow h_2)^2 \rho^2 N^2}{\mu^2 \pi^2 (2|s| \Leftrightarrow 1)^2} \end{aligned}$$

Also it is with $\mu^2 \rho^2 \leq h_2^2$ and (4.37)

$$\|\Psi_{\pm}\|_J^2 \frac{\mu^2}{\rho^2 N^2} = \frac{\mu^2}{2N} \Leftrightarrow \frac{N h_2^2 + h_2}{2\rho^2 N^2} \geq \frac{|h_2|}{2\rho^2 N^2}.$$

Then we get

$$\frac{1}{\|\Psi_{\pm}\|_J^2} |\psi_{\pm}^s|^2 \|U_s\|_1^4 \leq \frac{2(h_1 \Leftrightarrow h_2)^2 \rho^4 N^4}{|h_2| \pi^2} \frac{1}{(2|s| \Leftrightarrow 1)^2} = \frac{2\Theta \pi^2}{|h_2| (2|s| \Leftrightarrow 1)^2} \quad (\text{C.18})$$

(III) Further estimations

For the three terms in inequality (C.10) we get then with (4.66) and (4.67)

(1)

$$\begin{aligned} \sum_{|s| \geq K+2} \sum_{|t| \leq K} |d_t^s|^2 \|U_s\|_1^4 &= 2 \sum_{s=K+1}^{\infty} \sum_{t=-K}^{-1} |d_t^s|^2 \|U_s\|_1^4 + 2 \sum_{s=K+1}^{\infty} \sum_{t=1}^K |d_t^s|^2 \|U_s\|_1^4 \\ &\leq 14\Theta \sum_{s=K+1}^{\infty} \frac{1}{(2s \Leftrightarrow 1)^2} \sum_{t=1}^K \frac{1}{t^2} \end{aligned} \quad (\text{C.19})$$

(2)

$$\sum_{|s| \geq K+2} \frac{|\psi_+^s|^2 \|U_s\|_1^4}{\|\Psi_+\|_J^2} + \frac{|\psi_-^s|^2 \|U_s\|_1^4}{\|\Psi_-\|_J^2} \leq \frac{8\Theta \pi^2}{|h_2|} \sum_{s=K+2}^{\infty} \frac{1}{(2s \Leftrightarrow 1)^2} \quad (\text{C.20})$$

(3)

$$\begin{aligned} &\sum_{|t| \geq K+1} \sum_{|s| \leq K+1} |c_s^t|^2 \|V_t\|_2^4 = \\ &= 2 \sum_{t=-K-1}^{-\infty} \sum_{s=1}^{K+1} |c_s^t|^2 \|V_t\|_2^4 + 2 \sum_{t=K+1}^{\infty} \sum_{s=1}^{K-1} |c_s^t|^2 \|V_t\|_2^4 + \\ &\quad + 2|c_K^{K+1}| \|V_{K+1}\|_2^4 + 2|c_{K+1}^{K+1}| \|V_{K+1}\|_2^4 \\ &\leq \Theta \sum_{t=K+1}^{\infty} \frac{1}{t^2} \left[\sum_{s=1}^{K+1} \left(\frac{8}{(2s \Leftrightarrow 1)^2} + \frac{2}{s^2} \right) \right] + \frac{2\Theta}{(2K+1)^2 \eta^2} + \frac{2\Theta}{(2K \Leftrightarrow 1)^2} \end{aligned} \quad (\text{C.21})$$

Then substituting (C.19)-(C.21) into (C.10) yields

$$\begin{aligned} \Delta_K^2 &= \frac{\|\mathcal{A}\Phi_K \Leftrightarrow \alpha\Phi_K\|^2}{\|\Phi_K\|^2} \\ &\leq 3\chi^2 \left[\frac{28\Theta}{\rho^2 N} \sum_{s=K+1}^{\infty} \frac{1}{(2s \Leftrightarrow 1)^2} \sum_{t=1}^K \frac{1}{t^2} + \frac{8\Theta \pi^2}{|h_2|} \sum_{s=K+2}^{\infty} \frac{1}{(2s \Leftrightarrow 1)^2} \right] + \\ &\quad + \frac{2\chi^2}{\rho^2 N} \left[\Theta \sum_{t=K+1}^{\infty} \frac{1}{t^2} \left(\sum_{s=1}^{K+1} \frac{8}{(2s \Leftrightarrow 1)^2} + \sum_{s=1}^{K+1} \frac{2}{s^2} \right) + \frac{2\Theta}{(2K+1)^2 \eta^2} + \frac{2\Theta}{(2K \Leftrightarrow 1)^2} \right] \end{aligned}$$

In practice, we will choose K and then calculate the maximal possible value η_m for η from (C.11) and set, say, $\eta = 0.9\eta_m$. With the definition of Θ , the definitions $f(l) := \sum_{t=1}^l \frac{1}{t^2}$ and $g(l) := \sum_{t=1}^l \frac{1}{(2t-1)^2}$, the limits $\sum_{t \in \mathbb{N}} \frac{1}{t^2} = \frac{\pi^2}{6}$ and $\sum_{t \in \mathbb{N}} \frac{1}{(2t-1)^2} = \frac{\pi^2}{8}$ we then get

$$\begin{aligned} \Delta_K^2 \leq & 3\chi^2 \Theta \left[\frac{28f(K)}{\rho^2 N} \left(\frac{\pi^2}{8} \Leftrightarrow g(K) \right) + \frac{8\pi^2}{|h_2|} \left(\frac{\pi^2}{8} \Leftrightarrow g(K+1) \right) \right] + \\ & + \frac{2\chi^2 \Theta}{\rho^2 N} \left[\left(\frac{\pi^2}{6} \Leftrightarrow f(K) \right) \left(8g(K+1) + 2f(K+1) \right) + \frac{2}{(2K+1)^2 \eta^2} + \frac{2}{(2K \Leftrightarrow 1)^2} \right] \end{aligned} \quad (\text{C.22})$$

where $\Theta = (h_1 \Leftrightarrow h_2)^2 N^4 \rho^4 / \pi^4$. From above we see that

$$\lim_{K \rightarrow \infty} \Delta_K = 0$$

with the right hand side of inequality (C.22) as upper estimate for Δ_K , but note that this estimate is rather rough, also because of the estimations in (C.9).

C.2.3 The programme codes

The Galerkin method described in Section 4.4.2 is implemented in three steps:

- (I) Construction of the system matrix , - implemented in 'C'

```

input N, rho, h1, h2, K ;
calculate eigenvalues lambda_s, tau_s, s in Z*, and mu ,
    using Newton's method;
calculate the energy norms of the eigenvectors
    U_s, V_s, s in Z*, and Psi_pm ;
calculate the real and imaginary parts of the coefficients b_s^+, psi_+^s
    for s = 1, ..., K + 1 ;
calculate the coefficients c_t^s and d_t^s for s = K Leftrightarrow 1, ..., K + 1 and
    t = 1, ..., K ;
calculate 1/chi Delta_K ;
open files to store the data;
write the parameters N, rho, h1, h2, K and 1/chi Delta_K ,
    the eigenvalues mu, Leftrightarrow mu, (lambda_s)_s, (tau_t)_t ,
    and the matrices Re[1/(i chi), 2], Im[1/(i chi), 2]
    Re[1/(i chi), 3], Im[1/(i chi), 3], 1/(i chi), 5
    and 1/(i chi), 6 into the files;
close files;
end

```

The eigenvalues are calculated from the equations (4.22), (4.29), and (4.33) using Newton's method described in books on Numerical Analysis, for example in Stoer/Bulirsch[32].

- (II) Calculation of the eigenvalues $(\alpha_K^{(j)})_j$ of , - in 'Matlab'

```

input run, fromstep, tostep, stepsize, start;

```

```

call function 'importdata.m';
  open the files created in step (I);
  read the parameters  $N, \rho, h_1, h_2, K, 1/\varkappa\Delta_K$ ;
  read the eigenvalues  $\mu, \Leftrightarrow\mu, (\lambda_s), (\tau_s)$  ;
  read the matrices  $1/(i\varkappa), j, j = 1, 2, 3, 5, 6$ ;

  'paste' the matrices to  $\Omega = \begin{pmatrix} I & 1/(i\varkappa), 2 & 0 \\ 1/(i\varkappa), 3 & I & 1/(i\varkappa), 5 \\ 0 & 1/(i\varkappa), 6 & I \end{pmatrix}$ ;
for  $k = 1, \dots, \text{tostep} \Leftrightarrow \text{fromstep} + 1$ ;
  set  $\varkappa = \varkappa_0 + (k \Leftrightarrow 1)\text{stepsize}$  ;
  set  $\cdot = i\varkappa\Omega + \text{diag}(\text{Values})$ ;
  calculate condition number of  $\cdot$  ;
  calculate eigenvalues  $\alpha_K^{(j)}$  of  $\cdot$  ;
  open files to store the result;
  write the parameters  $N, \rho, h_1, h_2, K, \varkappa, \Delta_K$  and condition number;
  write the pairs  $[Re(\alpha_K^{(j)}), Im(\alpha_K^{(j)})]$ ;
  close files;
end loop;
end

```

The condition number of a matrix gives some information how accurate the numerical calculation is. It should not be too large. For more details see Stoer/Bulirsch[32].

(III) Illustration of the results in graphs - in 'Matlab'

```

input run, from, to;
for  $k = \text{from}, \dots, \text{to}-1$ ;
  call function 'readdata.m';
  open files created in step (II);
  read the files;
  close files;
  divide the eigenvalues into four groups:
     $G_1 = \{Im(\alpha) > \Delta_K\}$  'certain UHP'
     $G_2 = \{0 \leq Im(\alpha) \leq \Delta_K\}$  'uncertain UHP'
     $G_3 = \{0 \geq Im(\alpha) < \Leftrightarrow\Delta_K\}$  'uncertain LHP'
     $G_4 = \{\Leftrightarrow\Delta_K \geq Im(\alpha)\}$  'certain LHP'
  plot  $G_1$  as '.'; plot  $G_2$  as 'o';
  plot  $G_3$  as '*'; plot  $G_4$  as 'x';
  caption graphs with 'kappa', 'delta' and 'condition'
  and insert respective values;
end loop;
end

```

C.2.4 Some graphs

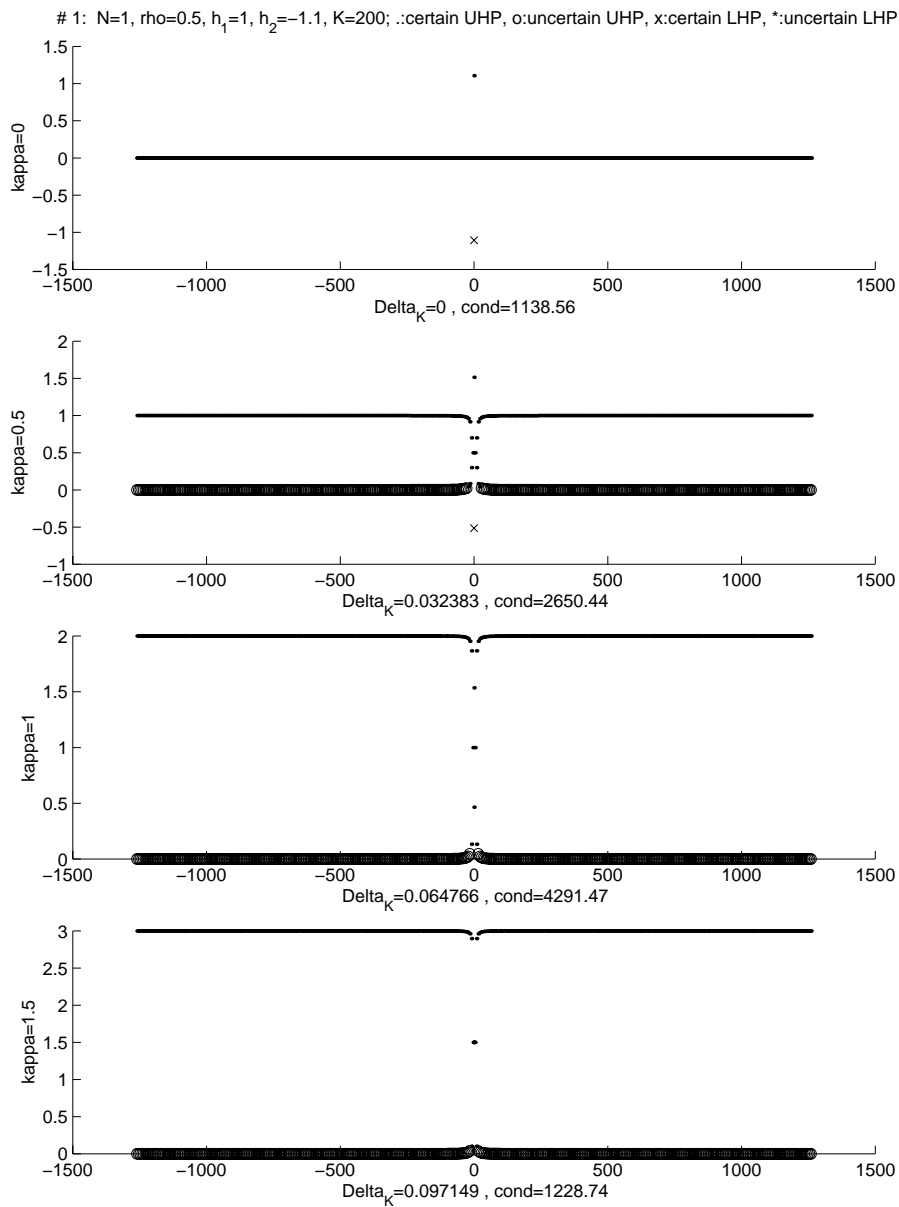


Figure C.2: $N = 1$, $\rho = 0.5$, $h_1 = 1$, $h_2 = \pm 1.1$, $K = 200$, $\kappa = 0, 0.5, 1, 1.5$

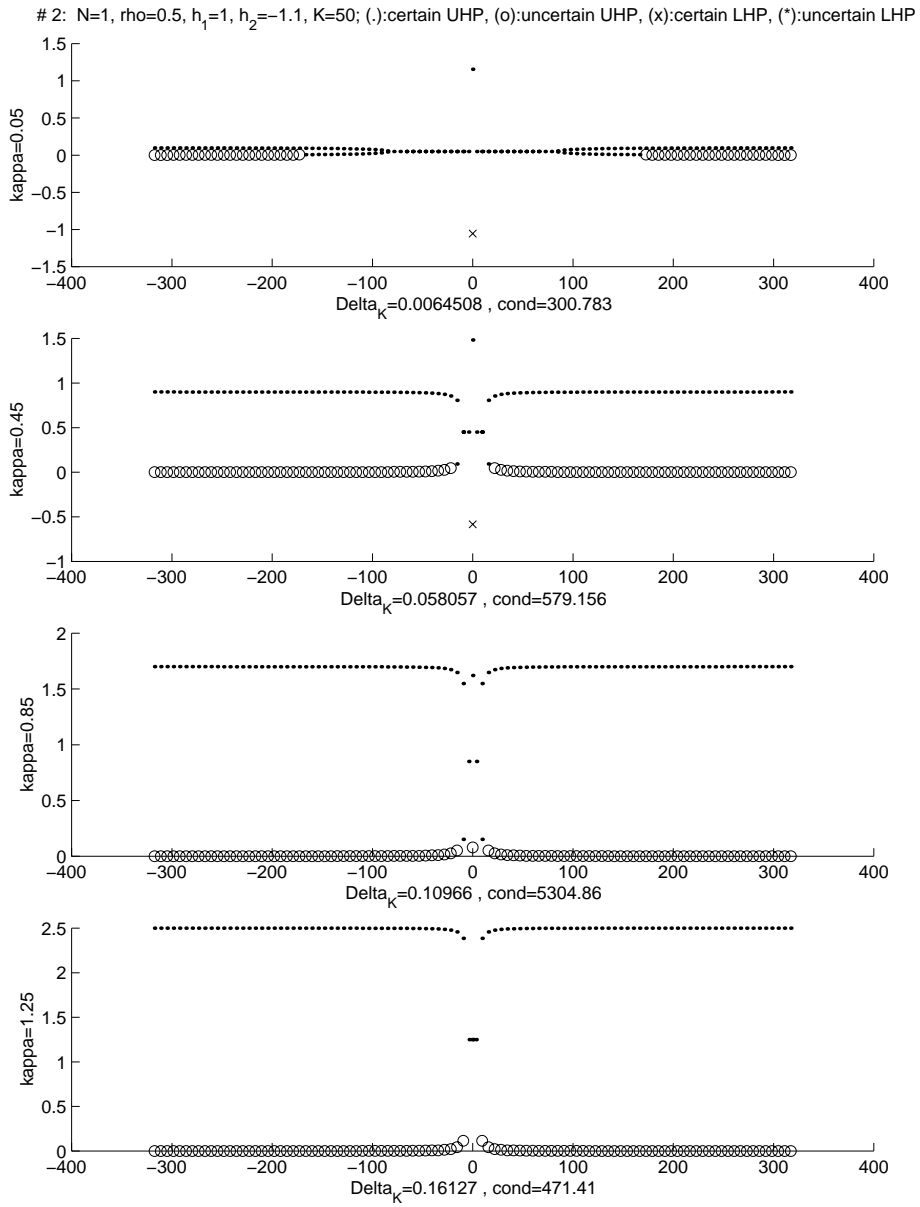


Figure C.3: $N = 1, \rho = 0.5, h_1 = 1, h_2 = \pm 1.1, K = 50, \kappa = 0.05, 0.45, 0.85, 1.25$

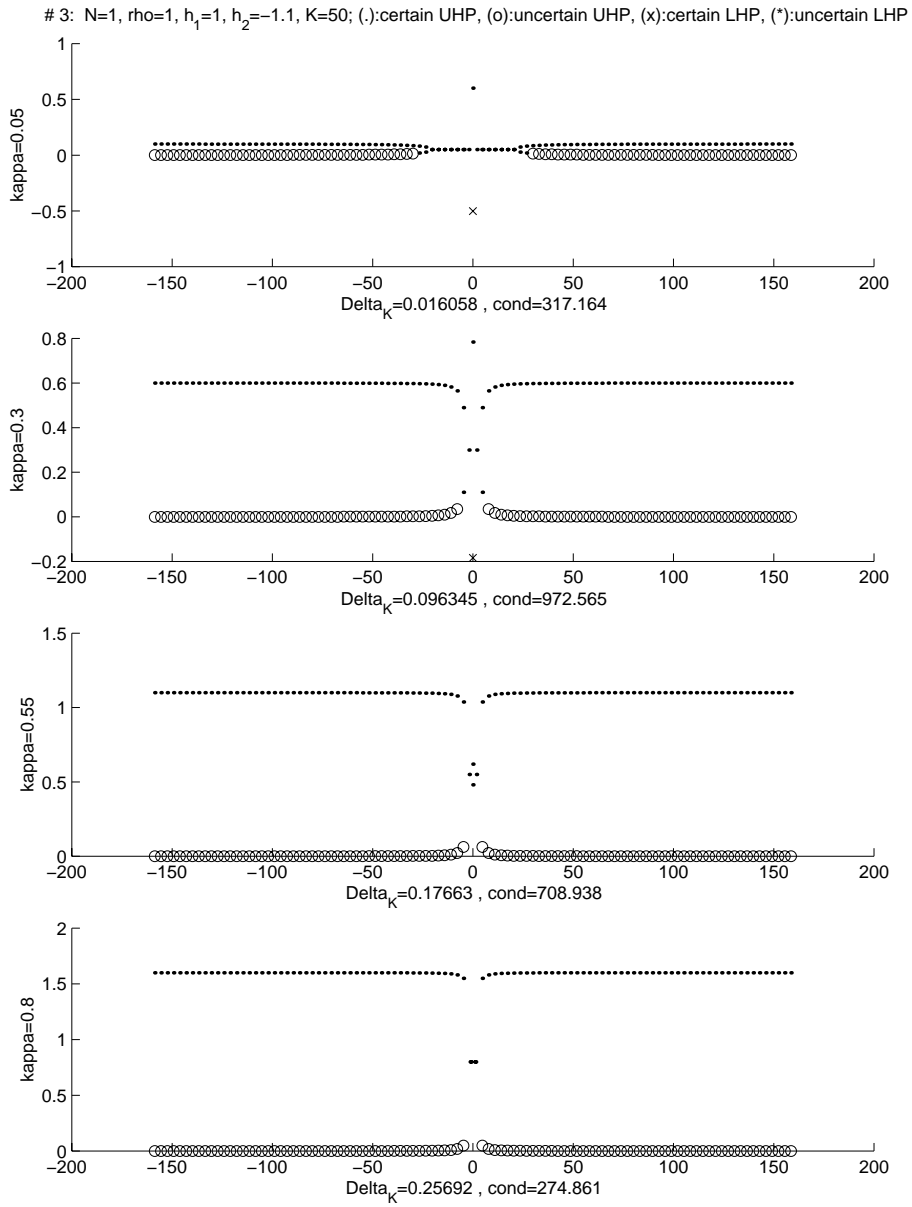


Figure C.4: $N = 1$, $\rho = 1$, $h_1 = 1$, $h_2 = -1.1$, $K = 50$, $\kappa = 0.05, 0.3, 0.55, 0.8$

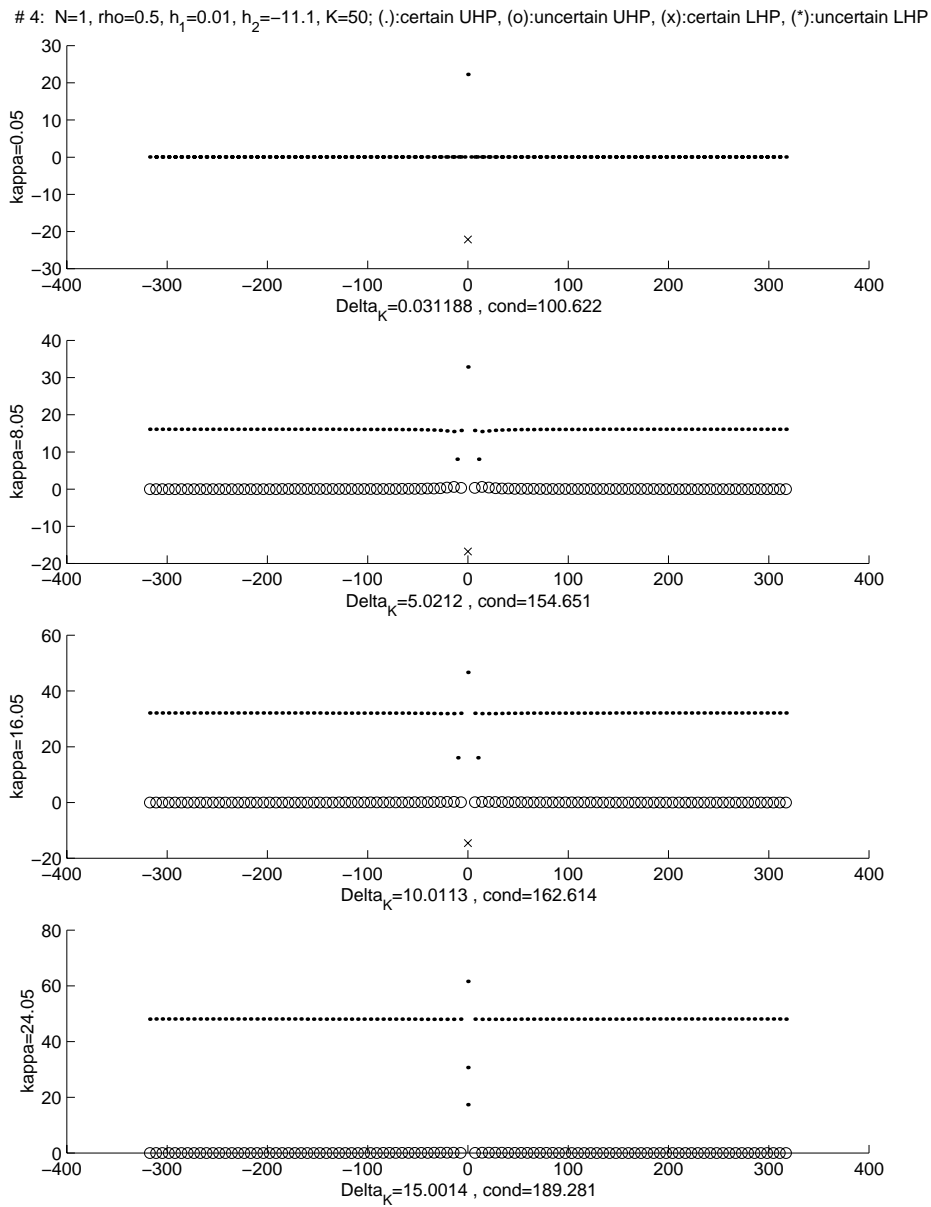


Figure C.5: $N = 1$, $\rho = 0.5$, $h_1 = 0.01$, $h_2 = -11.1$, $K = 50$, $\kappa = 0.05, 8.05, 16.05, 24.05$

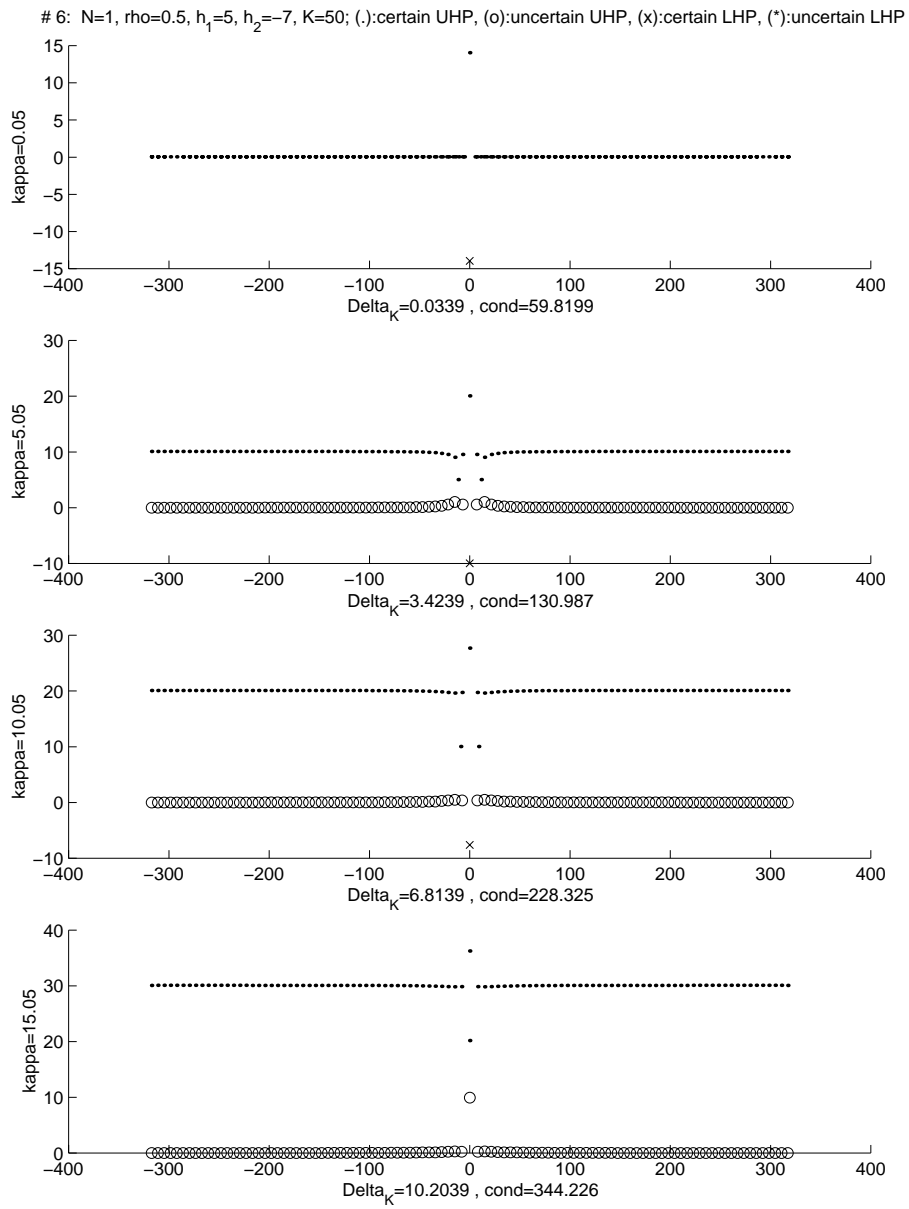


Figure C.6: $N = 1$, $\rho = 0.5$, $h_1 = 5$, $h_2 = -7$, $K = 50$, $\varkappa = 0.05, 5.05, 10.05, 15.05$

References

- [1] Akhiezer, N.I. and Glazman, I.M.: “Theory of Linear Operators in Hilbert Space”, *Two volumes bound as one*, Dover Publications 1993
- [2] Birman, M.S. and Solomjak, M.Z.: “Spectral Theory of Self-adjoint Operators in Hilbert Space”, *Mathematics and Its Applications*, D.Reidel Publishing Company 1987
- [3] Breiman, Leo: “Probability and Stochastic Processes - With a view toward Application”, Houghton Mifflin Company 1969
- [4] Bronstein, I.N. and Semendjajew, K.A.: “Taschenbuch der Mathematik”, 25. Ausgabe, Verlag Nauka, Moskau, und B.G.Teubner Verlagsgesellschaft, Stuttgart, 1991
- [5] Cheremshantsev, S.E.: “Theory of Scattering by a Brownian Particle”, *Proceedings of the Steklov Institute of Mathematics*, 1991, issue 2
- [6] Davies, E.B.: “One-parameter Semigroups”, Academic Press 1980
- [7] Diller, Antoni: “L^AT_EX, Line by Line”, John Wiley&Sons 1993
- [8] Dowson, H.R.: “Spectral Theory of Linear Operators”, Academic Press
- [9] Griego, R. and Hersh, R.: “Random evolutions, Markov chains and systems of partial differential equations”, *Proc. Nat. Acad. Sciences USA*, 62(1969) p305-308
- [10] Griego, R. and Hersh, R.: “Theory of random evolutions with applications to partial differential equations”, *Transactions Amer. Math. Soc.*, 156(1971) p405-418
- [11] Grimmett, G.R. and Stirzaker, D.R.: “Probability and Random Processes”, Oxford Science Publications 1993
- [12] Helson, H.: “Lectures on Invariant Spaces”, Academic Press 1964
- [13] Hersh, R.: “Random evolutions: A survey of results and problems”, *Rocky Mountain J. Math.*, Vol 4, no 3 (1974), p443-477
- [14] Hislop, P.D. and Sigal, I.M.: “Introduction to Spectral Theory - With Applications to Schroedinger Operators”, *Applied Mathematical Sciences Vol 113*, Springer 1996
- [15] Hoffman, K.: “Banach Spaces of Analytic Functions”, Dover Publications, 1988
- [16] Ivanov, S.A. and Pavlov, B.S.: “Carleson resonance series in the Regge problem”, *Izv. Akad. Nauk. SSSR, Ser. Mat.*, 42 (1978), 26-55

- [17] Kato, T.: “Perturbation Theory for Linear Operators”, *A Series of Comprehensive Studies in Mathematics 132*, Springer 1966
- [18] Keepler, M.: “Backward and forward random evolutions”, *Indiana Uni Math. J.*, Vol 24, no 10 (1975) p937-947
- [19] Keepler, M.: ”On Random Evolutions Induced by Countable State Space Markov Chains”, *Portugaliae Mathematica*, Vol. 37 Fasc. 3-4 (1978)
- [20] Keepler, M.: “On the sample path approach to random evolutions”, *Bull. Malaysian Math Soc*, Vol 5, ser 2 (1982), p79-86
- [21] Iohvidov, I.S. and Krein, M.G. and Langer, H.: “Introduction to the Spectral Theory of Operators in Spaces with an Indefinite Metric”, Akademie-Verlag, Berlin 1982
- [22] Lax, Peter D. and Phillips, Ralph S.: “Scattering Theory for Automorphic Functions”, *Annals of Mathematical Studies*, Princeton University Press, 1976
- [23] Mikhlin, S.G.: “Variational Methods in Mathematical Physics”, Pergamon Press 1964
- [24] Naimark, M.A.: “Linear Differential Operators, Part 1 and 2”, Frederik Ungar Publishing Co., 1967
- [25] Pavlov, B.S.: “Continuous Resonance Spectrum on a Non-physical Sheet”, *Soviet Math. Doklady*, Vol. 13 (1972), No. 5, p. 1417
- [26] Pavlov, B.S.: “Dilation Theory and Spectral Analysis of Nonselfadjoint Differential Operators”, *Amer. Math. Soc. Transl.*, (2) Vol. 115 (1980), p. 103
- [27] Pavlov, B.S.: “Spectral Analysis of a dissipative Singular Schrödinger Operator in Terms of a Functional Model”, *Encyclopaedia of Mathematical Sciences 65*, Springer 1995
- [28] Pavlov, B.S.: “Quantum Dynamics on a Markov Background and Irreversibility”, *Nonlinear Dynamics Chaotic and Complex Systems*, Proceedings of an International Conference held in Zkopane, Poland, Nov 1995 and Cambridge University Press 1997
- [29] Pazy, A.: “Semi-groups of Linear Operators and Applications to Partial Differential Equations”, *Applied Mathematical Sciences Vol. 44*, Springer 1983
- [30] Pinsky, M.: “Lectures on Random evolution”, World Scientific 1991
- [31] Riesz, Frigyes and Sz.-Nagy, Béla: “Functional Analysis”, Dover Publications 1990
- [32] Stoer, J. and Bulirsch, R.: “Introduction to Numerical Analysis”, Springer 1980
- [33] Strauss, W.A.: “Partial Differential Equations - An Introduction”, John Wiley&Sons 1992
- [34] Taylor, H.M. and Karlin, S.: “An Introduction To Stochastic Modelling”, Academic Press 1994

- [35] Tikhonov, A.N., Vasil'eva, A.B. and Sveshnikov, A.G.: "Differential Equations", Springer 1985
- [36] Ziemer, W.P.: "Weakly Differential Functions", Springer 1989

Abbreviations and Notations

w.l.o.g.	without limitation of generality
w.r.t.	with respect to
UHP	upper half plane, i.e. $\{z \in \mathbb{C} : \text{Im}(z) \geq 0\}$
LHP	lower half plane, i.e. $\{z \in \mathbb{C} : \text{Im}(z) < 0\}$
i	complex number i
	The attempt has been made to avoid the character i as an index.
$\bar{z} = \text{conj}(z)$	the complex conjugate of $z \in \mathbb{C}$
$+\sqrt{z}, z \in \mathbb{C}$	the root of z with the argument in the interval $[0, \pi)$
$\text{Pr}(x)$	the probability of an event X
$\text{Pr}(X Y)$	the probability of X under condition Y
$E[X]$	the expected value of a random variable X
$E[X Y]$	the conditional expected value
Q^t, p^t	the transpose of a matrix Q or vector p
$\sigma(A)$	the spectrum of an operator A
$\varrho(A)$	the resolvent set of an operator A
$\text{clos}_{\ \cdot\ }(X)$	the closure of a set X w.r.t. the topology generated by the norm $\ \cdot\ $
X^c	the set-theoretical complement of a set X
$\text{dist}(x, y)$	the distance between two elements x and y
$\text{dist}(x, X)$	the distance of an element x to a set X
$(x_n)_{n \in \mathbb{N}} \subset X$	a sequence of elements x_n in the set X
$r \rightarrow N+$	r approaches to 1 from above
$f(N+)$	the right limit $\lim_{r \rightarrow N+} f(r)$
$\text{supp}(f)$	the support of a function f
$\chi_{[0, N]}$	the characteristic function of the interval $[0, N]$
$C^k(\mathbb{R})$	the space of continuous functions on \mathbb{R} with continuous derivatives up to order k
$C_0^k(\mathbb{R})$	the space of continuous functions on \mathbb{R} with continuous and compactly supported derivatives up to order k
$\delta(x \Leftrightarrow N)$	the Dirac delta function, shifted to $x = N$
δ_{jk}	the Kronecker symbol