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Language-Theoretic Complexity of Disjunctive Sequences*

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Abstract

A sequence over an alphabet Σ is called *disjunctive* [13] if it contains all possible finite strings over Σ as its substrings. Disjunctive sequences have been recently studied in various contexts, e.g. [12, 9]. They abound in both category and measure senses [5].

In this paper we measure the complexity of a sequence $\mathbf{x}$ by the complexity of the language $P(\mathbf{x})$ consisting of all prefixes of $\mathbf{x}$. The languages $P(\mathbf{x})$ associated to disjunctive sequences can be arbitrarily complex. We show that for some disjunctive numbers x the language $P(\mathbf{x})$ is context-sensitive, but no language $P(\mathbf{x})$ associate to a disjunctive number can be context-free. We also show that computing a disjunctive number x by rationals corresponding to an infinite subset of $P(\mathbf{x})$ does not decrease the complexity of the procedure, i.e. if x is disjunctive, then $P(\mathbf{x})$ contains no infinite context-free language. This result reinforces, in a way, Chaitin's thesis [6] according to which *perfect* sets, i.e. sets for which there is no way to compute infinitely many of its members essentially better (simpler/quicker) than computing the whole set, do exist. Finally we prove the existence of the following language-theoretic complexity gap: There is no $\mathbf{x} \in \Sigma^\omega$ such that $P(\mathbf{x})$ is context-free but not regular. If the set of all finite substrings of a sequence $\mathbf{x} \in \Sigma^\omega$ is slender, then the set of all prefixes of $\mathbf{x}$ is regular, that is $P(\mathbf{x})$ is regular if and only if $S(\mathbf{x})$ is slender. The proofs essentially use some recent results concerning the complexity of languages containing a bounded number of strings of each length [15, 14, 11, 16].

1 Preliminaries

Let Σ be a finite set and denote by Σ^* and Σ^ω, respectively, the sets of all (finite) strings and (one-way infinite) sequences over Σ.

For $\mathbf{x}$ in Σ^ω we define the following two sets:

$$S(\mathbf{x}) = \{u \in \Sigma^* \mid \mathbf{x} = vu\mathbf{y},\ v \in \Sigma^*,\ \mathbf{y} \in \Sigma^\omega\},$$

and

$$P(\mathbf{x}) = \{u \in \Sigma^* \mid \mathbf{x} = u\mathbf{y},\ \mathbf{y} \in \Sigma^\omega\},$$

that is, $S(\mathbf{x})$ is the set of all finite substrings of $\mathbf{x}$, and $P(\mathbf{x})$ is the set of all finite prefixes of $\mathbf{x}$.

For a language $L \subseteq \Sigma^*$ define

$$S_f(L) = \{v \in \Sigma^* \mid uvw \in L,\ u, w \in \Sigma^*\}.$$

Note that S_f is similar to S, but for languages of finite strings rather than for infinite sequences. Similarly, we define

$$P_f(L) = \{u \in \Sigma^* \mid uw \in L,\ w \in \Sigma^*\}.$$

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For a finite string $u \in \Sigma^*$, $|u|$ denotes the length of u. For a language $L \subseteq \Sigma^*$, $card(L)$ denotes the cardinality of L.

Lemma 1.1. *For each* $\mathbf{x} \in \Sigma^\omega$, $S(\mathbf{x}) = S_f(P(\mathbf{x}))$.

Proof. If u belongs to $S(\mathbf{x})$, then $\mathbf{x} = vu\mathbf{y}$, for some $v \in \Sigma^*$ and $\mathbf{y} \in \Sigma^\omega$. So, vu is in $P(\mathbf{x})$ and, therefore, $u \in S_f(P(\mathbf{x}))$. Conversely, let $w \in S_f(P(\mathbf{x}))$, i.e. $uwv = z$, for some $u, v \in \Sigma^*$ and $z \in P(\mathbf{x})$. Since $z \in P(\mathbf{x})$, it follows that $\mathbf{x} = z\mathbf{x}'$, for some $\mathbf{x}' \in \Sigma^\omega$, that is $\mathbf{x} = uwv\mathbf{x}' = uw\mathbf{x}''$, where $\mathbf{x}'' = v\mathbf{x}'$. Consequently, $w \in S(\mathbf{x})$. □

For every language $L \subseteq \Sigma^*$ define the *density* function D_L by $D_L(n) = card(L \cap \Sigma^n)$, where Σ^n denotes the set of all strings of length n over Σ. If a language L has a constant density, i.e., $D_L = \mathrm{O}(1)$, then it is called a slender language, which was termed in [1]. The following results have been proven in [15].

Lemma 1.2. *A regular language R over Σ has a density $O(n^k), k \geq 0$ if and only if R can be represented as a finite union of regular expressions of the following form:*

$$xy_1^* z_1 \cdots y_t^* z_t,$$

where $x, y_1, z_1, \ldots, y_t, z_t \in \Sigma^*$ and $0 \leq t \leq k+1$.

Lemma 1.3. *Let R be a regular language, $R' = S_f(R)$ and let k be a non-negative integer. Then $D_R(n) = O(n^k)$ if and only if $D_{R'}(n) = O(n^k)$.*

Several of the subsequent proofs depend on the following result, which has been proved in [11] (see also [16]).

Lemma 1.4. *Let $L \subseteq \Sigma^*$ be a context-free language. Then L is slender, i.e., $D_L(n) = O(1)$, if and only if L is a finite union of languages of the form:*

$$\{u_1 u_2^i u_3 u_4^i u_5 \mid i \geq 0\},$$

where $u_1, u_2, u_3, u_4, u_5 \in \Sigma^*$.

2 How Complex Are Disjunctive Sequences?

A sequence $\mathbf{x} \in \Sigma^\omega$ is *disjunctive* [13] provided it contains all possible finite strings over Σ as its substrings, i.e. $S(\mathbf{x}) = \Sigma^*$.

At the top, disjunctive sequences $\mathbf{x}$ can be random, non-random but non-recursive, recursive, but arbitrarily complex. At the bottom, the complexity of a sequence $\mathbf{x}$ will be measured by the complexity of the language $P(\mathbf{x})$ consisting of all prefixes of $\mathbf{x}$; these languages can be context-sensitive, but not context-free.

Chaitin's Omega Number [7] is Borel normal in any base and, therefore, disjunctive in any base. More generally, by Theorem 3.6 in [3], every random sequence is Borel normal and, hence, disjunctive. All these sequences are non-recursive; they form a class of measure one [4]. Non-random and non-recursive disjunctive sequences have been constructed in [13].

Having disposed of the non-recursive case we turn our attention to recursive disjunctive sequences. First we rely on Rabin's Theorem (see, for instance, Theorem 3.5 in [2]) to construct arbitrarily complex recursive disjunctive sequences:

Theorem 2.1. *For every Blum space (φ_i, Φ_i) and for every recursive function B, a two-valued recursive function f can be effectively constructed such that, for every j with $\varphi_j = f$, one has $\Phi_j(n) > B(n)$, for almost all n.*

Theorem 2.2. *There exist recursive, arbitrarily complex, disjunctive sequences.*

Proof. Consider a primitive recursive enumeration ε of all non-empty strings over Σ and a recursive function B mapping positive integers into positive integers. Assume that B grows as fast as Ackermann's function [2]. Fix two letters, say σ_1, σ_2 in Σ and let f be a recursive function mapping positive integers into $\{\sigma_1, \sigma_2\}$ such that for every $\varphi_i = f$, $\Phi_i(n) > B(n)$, for almost all n. Construct the sequence

$$\mathbf{x} = f(1)\varepsilon(1)f(2)\varepsilon(2)\cdots f(n)\varepsilon(n)\cdots.$$

Clearly, $S(\mathbf{x}) = \Sigma^*$. Let $\mathbf{x}(n)$ be the prefix of length n of $\mathbf{x}$. Then for every integer $n \geq 2$,

$$f(n) = \psi(\mathbf{x}(\sum_{i=1}^{n-1} \mid \varepsilon(i) \mid +n)),$$

where $\psi(w)$ returns the last letter of the string w. Obviously, if $\varphi_j(n) = x_n$, (x_n is the nth term of $\mathbf{x}$), then $\Phi_j(n) > B(n)$, for almost all n. □

A natural way to produce a recursive disjunctive sequence is by concatenating, in some recursive order, all strings over a fixed alphabet. We can ask ourselves: Are there "simpler" ways to produce disjunctive sequences? We prove that the language of all prefixes of the sequence consisting of all strings over the binary alphabet arranged in quasi-lexicographical order is context-sensitive and show that this complexity is the best possible we can obtain.

Theorem 2.3. *There exists a disjunctive sequence $\mathbf{x} \in \Sigma^\omega$ such that $P(\mathbf{x})$ is context-sensitive.*

Proof. Let $\Sigma = \{0, 1\}$ and $\mathbf{x} = x_1x_2x_3x_4\cdots$, where $x_1 = 0$, $x_2 = 1$, $x_3 = 00$, $x_4 = 01, \ldots$ are all binary strings arranged in quasi-lexicographical order. Clearly, $S(\mathbf{x}) = \Sigma^*$. To prove that $P(\mathbf{x})$ is context-sensitive we will construct a deterministic linear-bounded automaton[1] $\mathcal{A}$ which accepts $P(\mathbf{x})$. The automaton has two tapes: a read-only input tape and a work tape. Initially, the input tape contains the input string with ♮ and \$ at the left end and the right end, respectively. Then $\mathcal{A}$ generates the strings $x_1, x_2, x_3, x_4, \ldots, x_n, \ldots$, one by one, on its work tape. At the same time, $\mathcal{A}$ checks whether the input is a catenation of $x_1, x_2, x_3, x_4, \ldots, x_{n-1}$ and a prefix of x_n, for some integer $n \geq 1$. If the above condition is fulfilled, then $\mathcal{A}$ accepts; otherwise, it rejects.

Here is a formal definition of the automaton $\mathcal{A} = (Q, \Sigma, \Gamma, \delta, q_0, B, \natural, \$, F)$:

$$\begin{array}{rcl} Q & = & \{q_0, q_1, q_2, q_3, q_4, f\} \text{ is the set of states;} \\ \Sigma & = & \{0, 1\} \text{ is the input alphabet;} \\ \Gamma & = & \Sigma \cup \{B, \natural, \$, \#\} \text{ is the tape alphabet;} \\ q_0 \in Q & & \text{is the initial state;} \\ B \in \Gamma & & \text{is the blank symbol;} \\ F = \{f\} & & \text{is the set of accepting states;} \end{array}$$

the transition function $\delta : Q \times \Gamma \times \Gamma \to Q \times \Gamma \times D \times D$ is denoted by $\delta(p, C_1, C_2) = (q, C_2', D_1, D_2)$, where p is the current state, C_1 is the symbol currently read by the head of the input tape and C_2 is the one by the head of the work tape, q is the next state, C_2' is the symbol written on the work tape, and D_1, D_2 are the moving directions of the input head and the work head, respectively, $D_1, D_2 \in \{L \text{ (left)}, R \text{ (right)}, \lambda \text{ (no move)}\}$.

The function δ is defined as follows:

[1] The automaton constructed in the proof actually uses only logarithmic space.

$$
\begin{aligned}
&\delta(q_0, \natural, B) = (q_0, \#, R, R),\\
&\delta(q_0, \$, B) = f,\\
&\delta(q_0, 0, B) = (q_1, 0, \lambda, R),\\
&\delta(q_1, 0, B) = (q_2, 0, \lambda, L),\\
&\delta(q_2, X, X) = (q_2, X, R, L),\\
&\delta(q_2, X, Y) = \text{ reject if } X \neq Y,\\
&\delta(q_2, \$, X) = f,\\
&\delta(q_2, X, \#) = (q_3, \#, \lambda, R),\\
&\delta(q_3, X, 0) = (q_4, 1, \lambda, R),\\
&\delta(q_3, X, 1) = (q_3, 0, \lambda, R),\\
&\delta(q_3, X, B) = (q_4, 0, \lambda, R),\\
&\delta(q_4, X, Y) = (q_3, Y, \lambda, R),\\
&\delta(q_4, X, B) = (q_2, B, \lambda, L),
\end{aligned}
$$

where $X, Y \in \{0, 1\}$. All undefined transitions result in rejection. □

Corollary 2.4. *There exist infinitely many disjunctive sequences $\mathbf{x} \in \Sigma^\omega$ such that $P(\mathbf{x})$ is context-sensitive.*

Proof. Consider the sequence of strings $(x_i)_{i\geq 1}$ used in the proof of Theorem 2.3. Let $\mathbf{y}(i) = x_i x_{i+1} \cdots$, $i \geq 1$. Clearly, for all $i \geq 1$, $S(\mathbf{y}(i)) = \Sigma^*$ and $P(\mathbf{y}(i))$ is deterministic context-sensitive by virtue of a proof similar to that of Theorem 2.3. □

Theorem 2.5. *For every sequence $\mathbf{x} \in \Sigma^\omega$, if $P(\mathbf{x})$ is regular, then $S(\mathbf{x})$ is regular, and both of them are slender; more precisely,*

$$D_{P(\mathbf{x})}(n) = 1, \text{ and } D_{S(\mathbf{x})}(n) = O(1).$$

Proof. Let $\mathbf{x} \in \Sigma^\omega$. If $P(\mathbf{x})$ is regular, then, by Lemma 1.1, $S(\mathbf{x}) = S_f(P(\mathbf{x}))$ and, thus, $S(\mathbf{x})$ is also regular. Clearly, $D_{P(\mathbf{x})}(n) = 1$, for all integer $n \geq 0$. So, by Lemma 1.3, $D_{S(\mathbf{x})}(n) = \mathrm{O}(1)$. □

Theorem 2.6. *For every sequence $\mathbf{x} \in \Sigma^\omega$, if $P(\mathbf{x})$ is context-free, then $S(\mathbf{x})$ is context-free, and*

$$D_{P(\mathbf{x})}(n) = 1, \text{ and } D_{S(\mathbf{x})}(n) = O(n).$$

Proof. Let $\mathbf{x} \in \Sigma^\omega$ such that $P(\mathbf{x})$ is context-free. By Lemma 1.1, $S(\mathbf{x}) = S_f(P(\mathbf{x}))$. Then it is clear that $S(\mathbf{x})$ is also context-free. Again, $D_{P(\mathbf{x})}(n) = 1$. By Lemma 1.4, $P(\mathbf{x})$ can be described as a finite union of terms of the form $uv^i wx^i y$, i.e.

$$P(\mathbf{x}) = \bigcup_{j=1}^{k} \{u_j v_j^{i_j} w_j x_j^{i_j} y_j \mid i_j \geq 0\},$$

for some integer constant $k \geq 0$. Let R be the regular language

$$R = \bigcup_{j=1}^{k} u_i v_j^* w_j x_j^* y_j.$$

Clearly, $P(\mathbf{x}) \subset R$ and, thus, $S_f(P(\mathbf{x})) \subset S_f(R)$. By Lemma 1.2, $D_R(n) = \mathrm{O}(n)$. Thus, $D_{S_f(R)}(n) = \mathrm{O}(n)$, by Lemma 1.3. Finally, $D_{S(\mathbf{x})}(n) = \mathrm{O}(n)$ as $S(\mathbf{x}) = S_f(P(\mathbf{x})) \subset S_f(R)$. □

Corollary 2.7. *For every disjunctive sequence $\mathbf{x} \in \Sigma^\omega$, $P(\mathbf{x})$ is not context-free.*

Proof. Let $\mathbf{x}$ be in Σ^ω such that $S(\mathbf{x}) = \Sigma^*$. Assume that $P(\mathbf{x})$ is context-free. By Theorem 2.6, $D_{S(\mathbf{x})}(n) = \mathrm{O}(n)$. But, $D_{\Sigma^*}(n) = card(\Sigma^n)$ is exponential. So, $S(\mathbf{x}) \neq \Sigma^*$, which is a contradiction. $\square$

Corollary 2.8. *If $\mathbf{x} \in \Sigma^\omega$ is disjunctive, then $P(\mathbf{x})$ contains no infinite context-free language.*

Proof. Let P' be an infinite subset of $P(\mathbf{x})$. It is clear that $P(\mathbf{x})$ is the set of all prefixes of P'. If P' would be context-free, then it is easy to show that $P(\mathbf{x})$ itself would be context-free, which contradicts Corollary 2.7. $\square$

Let $x \in [0, 1]$ be such that its b-expansion $x_1x_2 \cdots x_n \cdots$ is disjunctive. The sequence of rationals $r_n = \sum_{i=1}^{n} x_i b^{-i}$ converges to x as $|x - r_n| \leq b^{-n}$. Consider now a function f, from positive integers to positive integers, with an infinite range. The sequence $\{r_{f(n)}\}$ is still convergent to x, and by Corollary 2.7, computing the approximations $\{r_{f(n)}\}$ is as difficult as computing the approximations $\{r_n\}$. This is another example supporting Chaitin's thesis [6] concerning *perfect* sets.

3 A Language-Theoretic Complexity Gap and Others

In this section we show that there is no $\mathbf{x} \in \Sigma^\omega$ such that $P(\mathbf{x})$ is context-free and not regular. Some of the results in the previous section can be proved immediately using this result. However, we keep those direct proofs because we feel they are simple and interesting by themselves.

We then conclude this section by showing that if the set of all finite substrings of a sequence $\mathbf{x} \in \Sigma^\omega$ is slender, then the set of all prefixes of $\mathbf{x}$ is regular. In view of Theorem 2.5 it follows that, for any $\mathbf{x} \in \Sigma^\omega$, $P(\mathbf{x})$ is regular if and only if $S(\mathbf{x})$ is slender.

Our first proof makes use of the following result proved in [8]. For positive integers i, j, denote by (i, j) the greatest common divisor of i and j.

Lemma 3.1. *Let $u, v \in \Sigma^*$. Then $u = w^m$ and $v = w^n$ for some $w \in \Sigma^*$, $m, n \geq 0$, if and only if there exist $p, q \geq 0$ so that u^p and v^q contain a common prefix of length $|u| + |v| - (|u|, |v|)$.*

Theorem 3.2. *Let $\mathbf{x} \in \Sigma^\omega$. If $P(\mathbf{x})$ is context-free, then $P(\mathbf{x})$ is regular.*

Proof. Let $\mathbf{x} \in \Sigma^\omega$ such that $P(\mathbf{x})$ is context-free. Then by Lemma 1.4, $P(\mathbf{x})$ can be represented as a finite union of terms of the form $u_1u_2^iu_3u_4^iu_5$, for some $u_1, \ldots, u_5 \in \Sigma^*$. Let $u_1u_2^iu_3u_4^iu_5$ be one of such terms of $P(\mathbf{x})$, and let $P_0 = \{u_1u_2^iu_3u_4^iu_5 \mid i \geq 0\}$. It is clear that $P_f(P_0) = P(\mathbf{x})$.

If $u_2 = \lambda$ or $u_4 = \lambda$, then P_0 is regular. Thus, $P(\mathbf{x}) = P_f(P_0)$ is regular. Now we assume that both $u_2 \neq \lambda$ and $u_4 \neq \lambda$. Let s and t be two arbitrary integers such that $s \geq |u_2u_4| + |u_2|$ and $t \geq |u_2^su_3u_4^su_5|$. Let $v_1 = u_1u_2^su_3u_4^su_5$ and $v_2 = u_1u_2^tu_3u_4^tu_5$. Then v_1 is a prefix of $u_1u_2^t$. Also, $u_3u_4^s$ is a prefix of u_2^{t-s}. Then there exists a decomposition $u_4 = u_{41}u_{42}$ such that $u_3u_4^{j_0}u_{41} = u_2^{k_0}$, for some constant $j_0 \leq |u_2|$, and $u_3u_4^s = u_2^{k_0}(u_{42}u_{41})^{s-j_0-1}u_{42}$. Let $\overline{u}_4 = u_{42}u_{41}$ and $s' = s - j_0 - 1$. Then $\overline{u}_4^{s'}$ and $u_2^{t-s-k_0}$ have a common prefix of length at least $|u_2| + |\overline{u}_4| - (|u_2|, |\overline{u}_4|)$. By Lemma 3.1, we have $u_2 = w^m$ and $\overline{u}_4 = w^n$ for some $w \in \Sigma^*$ and $m, n > 0$. Then for every $i > j_0$, we have

$$\begin{aligned} u_1u_2^iu_3u_4^iu_5 &= u_1u_2^iu_2^{k_0}\overline{u}_4{}^{i-j_0-1}u_{42}u_5 \\ &= u_1u_2^{i+k_0}\overline{u}_4{}^{i-j_0-1}u_{42}u_5 \\ &= u_1w^{(m+n)(i-1)}w^{mk_0+m-nj_0}u_{42}u_5. \end{aligned}$$

Let $\overline{u}_5 = w^{mk_0+m-nj_0}u_{42}u_5$ and $u = w^{m+n}$. Then $u_1u_2^iu_3u_4^iu_5 = u_1u^{i-1}\overline{u}_5$, for every $i > j_0$. So,

$$P_0 = \{u_1u_2^iu_3u_4^iu_5 \mid i \leq j_0\} \cup \{u_1w^{j-1}\overline{u}_5 \mid j > j_0\}.$$

Obviously, P_0 is regular. Therefore, $P(\mathbf{x}) = P_f(P_0)$ is regular too. $\square$

Theorem 3.3. *For any $\mathbf{x} \in \Sigma^\omega$, if $S(\mathbf{x})$ is slender, i.e., $D_{S(\mathbf{x})}(n) = O(1)$, then $P(\mathbf{x})$ is regular.*

Proof. Let $\mathbf{x} \in \Sigma^\omega$ such that $D_{S(\mathbf{x})}(n) \leq c$, for some constant $c > 0$. Assume that $P(\mathbf{x})$ is not regular. Let $R_{P(\mathbf{x})}$ be the right-invariant relation defined for all $u, v \in \Sigma^*$, by $(u, v) \in R_{P(\mathbf{x})}$ if for any $w \in \Sigma^*$, $uw \in P(\mathbf{x})$ if and only if $vw \in P(\mathbf{x})$. Denote by $[u]$ the equivalence class of $R_{P(\mathbf{x})}$ that contains u. By Myhill-Nerode Theorem [10], $P(\mathbf{x})$ contains words from infinitely many equivalence classes of $R_{P(\mathbf{x})}$. Let $U = \{u_1, u_2, \ldots, u_t\}$, for some $t > c$, such that $[u_i] \neq [u_j]$, for each pair $u_i, u_j \in U$ and $i \neq j$. Note that for each u_i, $1 \leq i \leq t$, there is exactly one word v for each length n such that $u_i v \in P(\mathbf{x})$. Denote by $v_i^{(n)}$ the word of length n such that $u_i v_i^{(n)} \in P(\mathbf{x})$.

Let m_1 be the smallest integer such that $v_j^{(m_1)} \neq v_k^{(m_1)}$, for some $j, k \in \{1, \ldots, t\}$. If all $v_i^{(m_1)}$, $1 \leq i \leq t$, are pairwise distinct, then we have t different words of length m_1 in $S(\mathbf{x})$. Since $t > c$, this is a contradiction. Otherwise, there exist j' and k' such that $v_{j'}^{(m_1)} = v_{k'}^{(m_1)}$. Then there exists $m_2 > m_1$ such that $v_{j'}^{(m_2)} \neq v_{k'}^{(m_2)}$. Note that it is clear that $v_j^{(m_2)} \neq v_k^{(m_2)}$ since $v_j^{(m_1)} \neq v_k^{(m_1)}$ and $m_2 > m_1$. By repeating this process for at most $t - 1$ times, we can obtain an integer m such that the strings $v_i^{(m)}$, $1 \leq i \leq t$ are pairwise distinct. Therefore, there are at least t distinct words of length m in $S(\mathbf{x})$. This is a contradiction. $\square$

From Theorem 2.5 and Theorem 3.3, we have the following two corollaries:

Corollary 3.4. *Let $\mathbf{x} \in \Sigma^\omega$. Then $P(\mathbf{x})$ is regular if and only if $S(\mathbf{x})$ is slender, i.e., $D_{S(\mathbf{x})} = O(1)$.*

Corollary 3.5. *For any $\mathbf{x} \in \Sigma^\omega$, the density function of $S(\mathbf{x})$ is not bounded by a constant if and only if $P(\mathbf{x})$ is not context-free.*

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