Primitive ideals in Hopf algebra extensions

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Hopf algebras

- (Finite dimensional) associative algebra H over field K whose dual H^* is also a Hopf algebra.
- Examples: group algebras KG (simplest), enveloping algebras $u(L)$, "quantum groups".
- Full structure $(H, m, \iota, \Delta, \varepsilon, S)$ where Δ, m are (co)multiplication, ε, ι (co)unit and S antipodal map.

Hopf algebra actions

- Hopf algebras naturally act on algebras with a product rule derived from Δ. Generalizes actions by automorphisms ($H = KG$) and derivations ($H = u(L)$).
- Basic setup: H acts on A with algebra of invariants $R = A^H$. Study relationship between R and A (general form of invariant and Galois theory).
- Examples/applications: Galois theory; quantum groups at roots of 1; smash products $R \# H^*$.
- Given $R \subset A$ can form dual extension $A \subset A \# H$.

Faithfully flat Galois extensions

- General extensions $R \subset A$ can behave badly even for $H = KG$ — further hypotheses required.
- Schneider: the class of faithfully flat Galois (ffG) H-extensions has nice stability properties and is sufficiently large to prove decent results.
- $R \subset A$ is ffG if and only if $V \mapsto V^{\uparrow} = A \otimes_R V$ is a category equivalence between R-mod and $A\#H$-mod.
- For most questions it suffices to consider smash products $R\#H^*$.

Primitive ideals

- Paper deals with primitive ideals (annihilators of simple modules) in ffG extensions. First step to proving generalizations of group algebra results.
- I reduce results of Montgomery and Schneider on prime ideals (Krull relations of lying over, going up, incomparability, etc) to the primitive case.
- Interesting module properties arise: finite induction property (${}_RV$ simple $\Rightarrow$ ${}_AV^{\uparrow}$ finite length), semisimple induction property (as above but ${}_AV^{\uparrow}$ also semisimple). They imply some Krull relations.

Results and conjectures

- H^* pointed (every irrep 1-dimensional): finite induction property, 5/6 Krull relations hold.
- H^* semisimple: satisfies everything if $\dim H < 60$.
- Conjecture: every H satisfies finite induction property and all Krull relations.
- Conjecture: semisimple induction property holds iff H^* semisimple.

Example

Let G be a connected, simply connected, semisimple complex algebraic group, let l be an odd integer and let ε be a primitive l-th root of 1. The quantum coordinate (Hopf) algebra $A = \mathcal{O}_\varepsilon(G)$ has a Hopf subalgebra $R \cong \mathcal{O}(G)$. The extension $R \subset A$ is ffG with respect to $H = (A/AR^+)^*$ and H is pointed.

Thus Spec(A) and Prim(A) are easily describable in terms of the (known) spectra of R.