Asymptotics of Coefficients of Multivariate Generating Functions

Mark C. Wilson, University of Auckland

Canberra, 2007-09

References

- mvGF site: www.cs.auckland.ac.nz/˜mcw/Research/mvGF/ .
- P. Flajolet and R. Sedgewick, *Analytic Combinatorics*, drafts at algo.inria.fr/flajolet/Publications/ .
- A. Odlyzko, survey on Asymptotic Enumeration Methods in *Handbook of Combinatorics*, Elsevier 1995, available from www.dtc.umn.edu/˜odlyzko/doc/enumeration.html.
- E. Bender, survey on Asymptotic Enumeration, SIAM Review 16:485-515, 1974.

Notation

- We use boldface to denote a multi-index: $\mathbf{z} = (z_1, \ldots, z_d)$, $\mathbf{r} = (r_1, \ldots, r_d)$. Similarly $\mathbf{z}^{\mathbf{r}} = z_1^{r_1} \ldots z_d^{r_d}$.

Notation

- We use boldface to denote a multi-index: $\mathbf{z} = (z_1, \ldots, z_d)$, $\mathbf{r} = (r_1, \ldots, r_d)$. Similarly $\mathbf{z}^{\mathbf{r}} = z_1^{r_1} \ldots z_d^{r_d}$.
- A (multivariate) sequence is a function $a : \mathbb{N}^d \to \mathbb{C}$ for some fixed d. Usually write $a_{\mathbf{r}}$ instead of $a(\mathbf{r})$.

Notation

- We use boldface to denote a multi-index: $\mathbf{z} = (z_1, \ldots, z_d)$, $\mathbf{r} = (r_1, \ldots, r_d)$. Similarly $\mathbf{z}^{\mathbf{r}} = z_1^{r_1} \ldots z_d^{r_d}$.
- A (multivariate) sequence is a function $a : \mathbb{N}^d \to \mathbb{C}$ for some fixed d. Usually write $a_{\mathbf{r}}$ instead of $a(\mathbf{r})$.
- The combinatorial case is when all $a_{\mathbf{r}} \geq 0$.

Notation

- We use boldface to denote a multi-index: $\mathbf{z} = (z_1, \ldots, z_d)$, $\mathbf{r} = (r_1, \ldots, r_d)$. Similarly $\mathbf{z}^\mathbf{r} = z_1^{r_1} \ldots z_d^{r_d}$.
- A (multivariate) sequence is a function $a : \mathbb{N}^d \to \mathbb{C}$ for some fixed d. Usually write $a_\mathbf{r}$ instead of $a(\mathbf{r})$.
- The combinatorial case is when all $a_\mathbf{r} \geq 0$.
- The generating function of the sequence is the formal power series $F(\mathbf{z}) = \sum_\mathbf{r} a_\mathbf{r} \mathbf{z}^\mathbf{r}$.

Notation

- We use boldface to denote a multi-index: $\mathbf{z} = (z_1, \dots, z_d)$, $\mathbf{r} = (r_1, \dots, r_d)$. Similarly $\mathbf{z}^{\mathbf{r}} = z_1^{r_1} \dots z_d^{r_d}$.
- A (multivariate) sequence is a function $a : \mathbb{N}^d \to \mathbb{C}$ for some fixed d. Usually write $a_{\mathbf{r}}$ instead of $a(\mathbf{r})$.
- The combinatorial case is when all $a_{\mathbf{r}} \geq 0$.
- The generating function of the sequence is the formal power series $F(\mathbf{z}) = \sum_{\mathbf{r}} a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}}$.
- If the series converges in a neighbourhood of $\mathbf{0} \in \mathbf{C}^d$, then F defines an analytic function there.

Cauchy integral formula approach

- Let U be the open disc of convergence, $\partial \mathrm{U}$ its boundary, C a circle centred at 0, inside U. Then

$$a_r = \frac{1}{2\pi i} \int_C z^{-r-1} F(z)\, dz.$$

Cauchy integral formula approach

- Let U be the open disc of convergence, $\partial\,\mathrm{U}$ its boundary, C a circle centred at 0, inside U. Then

$$a_r = \frac{1}{2\pi i}\int_C z^{-r-1}F(z)\,dz.$$

- Let ρ be the radius of U. The exponential growth rate is $1/\rho$:

$$\limsup \frac{1}{r}\log|a_r| = -\log|\rho|.$$

Cauchy integral formula approach

- Let U be the open disc of convergence, $\partial\,\mathrm{U}$ its boundary, C a circle centred at 0, inside U. Then

$$a_r = \frac{1}{2\pi i}\int_C z^{-r-1} F(z)\,dz.$$

- Let ρ be the radius of U. The exponential growth rate is $1/\rho$:

$$\limsup \frac{1}{r}\log|a_r| = -\log|\rho|.$$

- Suppose that $\rho < \infty$. Then in the combinatorial case

Further analysis depends on the type of singularity.

Cauchy integral formula approach

- Let U be the open disc of convergence, $\partial\,\mathrm{U}$ its boundary, C a circle centred at 0, inside U. Then

$$a_r = \frac{1}{2\pi i}\int_C z^{-r-1}F(z)\,dz.$$

- Let ρ be the radius of U. The exponential growth rate is $1/\rho$:

$$\limsup \frac{1}{r}\log|a_r| = -\log|\rho|.$$

- Suppose that $\rho < \infty$. Then in the combinatorial case
  - (Vivanti-Pringsheim) $z = \rho$ is a singularity of F;

Further analysis depends on the type of singularity.

Cauchy integral formula approach

- Let U be the open disc of convergence, $\partial\,\mathrm{U}$ its boundary, C a circle centred at 0, inside U. Then

$$a_r = \frac{1}{2\pi i}\int_C z^{-r-1}F(z)\,dz.$$

- Let ρ be the radius of U. The exponential growth rate is $1/\rho$:

$$\limsup \frac{1}{r}\log|a_r| = -\log|\rho|.$$

- Suppose that $\rho < \infty$. Then in the combinatorial case
  - (Vivanti-Pringsheim) $z = \rho$ is a singularity of F;
  - If F is aperiodic, $z = \rho$ is the only singularity on $\partial\,\mathrm{U}$.

  Further analysis depends on the type of singularity.

From singularities to asymptotic expansions

- There are standard methods for dealing with each type of singularity:

From singularities to asymptotic expansions

- There are standard methods for dealing with each type of singularity:
  - If ρ is large (essential), use the saddle point method. Move contour so the maximum modulus of the integrand is minimized.

From singularities to asymptotic expansions

- There are standard methods for dealing with each type of singularity:
  - If ρ is large (essential), use the saddle point method. Move contour so the maximum modulus of the integrand is minimized.
  - Otherwise, if F can be continued past $\partial\,\mathrm{U}$:

From singularities to asymptotic expansions

- There are standard methods for dealing with each type of singularity:
  - If ρ is large (essential), use the saddle point method. Move contour so the maximum modulus of the integrand is minimized.
  - Otherwise, if F can be continued past $\partial \mathrm{U}$:
    - if ρ is algebraic/logarithmic, use singularity analysis (Flajolet-Odlyzko 1990);

From singularities to asymptotic expansions

- There are standard methods for dealing with each type of singularity:
  - If ρ is large (essential), use the saddle point method. Move contour so the maximum modulus of the integrand is minimized.
  - Otherwise, if F can be continued past $\partial\,\mathrm{U}$:
    - if ρ is algebraic/logarithmic, use singularity analysis (Flajolet-Odlyzko 1990);
    - if ρ is a pole, use the residue theorem (below);

From singularities to asymptotic expansions

- There are standard methods for dealing with each type of singularity:
  - If ρ is large (essential), use the saddle point method. Move contour so the maximum modulus of the integrand is minimized.
  - Otherwise, if F can be continued past $\partial\,\mathrm{U}$:
    - if ρ is algebraic/logarithmic, use singularity analysis (Flajolet-Odlyzko 1990);
    - if ρ is a pole, use the residue theorem (below);
    - if F is rational, can also use partial fraction decomposition.

From singularities to asymptotic expansions

- There are standard methods for dealing with each type of singularity:
  - If ρ is large (essential), use the saddle point method. Move contour so the maximum modulus of the integrand is minimized.
  - Otherwise, if F can be continued past $\partial\,\mathrm{U}$:
    - if ρ is algebraic/logarithmic, use singularity analysis (Flajolet-Odlyzko 1990);
    - if ρ is a pole, use the residue theorem (below);
    - if F is rational, can also use partial fraction decomposition.
  - If $\partial\,\mathrm{U}$ is a natural boundary, use Darboux' method or circle method or . . . .

Example: derangements

- Consider $F(z) = e^{-z}/(1-z)$, the GF for derangements. There is a single pole at $z = 1$.

Example: derangements

- Consider $F(z) = e^{-z}/(1-z)$, the GF for derangements. There is a single pole at $z = 1$.
- Using a circle of radius $1+\varepsilon$ we obtain, by the residue theorem,

$$a_r = \frac{1}{2\pi i}\int_{C_{1+\varepsilon}} z^{-r-1}F(z)\,dz - \mathrm{Res}(z^{-r-1}F(z); z = 1).$$

Example: derangements

- Consider $F(z) = e^{-z}/(1-z)$, the GF for derangements. There is a single pole at $z = 1$.
- Using a circle of radius $1 + \varepsilon$ we obtain, by the residue theorem,

$$a_r = \frac{1}{2\pi i} \int_{C_{1+\varepsilon}} z^{-r-1} F(z)\, dz - \operatorname{Res}(z^{-r-1}F(z); z = 1).$$

- The integral is $O((1+\varepsilon)^{-r})$ while the residue equals $-e^{-1}$.

Example: derangements

- Consider $F(z) = e^{-z}/(1-z)$, the GF for derangements. There is a single pole at $z = 1$.
- Using a circle of radius $1+\varepsilon$ we obtain, by the residue theorem,

$$a_r = \frac{1}{2\pi i} \int_{C_{1+\varepsilon}} z^{-r-1} F(z)\, dz - \mathrm{Res}(z^{-r-1} F(z); z = 1).$$

- The integral is $O((1+\varepsilon)^{-r})$ while the residue equals $-e^{-1}$.
- Thus $[z^r]F(z) \sim e^{-1}$ as $r \to \infty$.

Example: derangements

- Consider $F(z) = e^{-z}/(1-z)$, the GF for derangements. There is a single pole at $z = 1$.
- Using a circle of radius $1 + \varepsilon$ we obtain, by the residue theorem,

$$a_r = \frac{1}{2\pi i} \int_{C_{1+\varepsilon}} z^{-r-1} F(z)\, dz - \mathrm{Res}(z^{-r-1}F(z); z = 1).$$

- The integral is $O((1+\varepsilon)^{-r})$ while the residue equals $-e^{-1}$.
- Thus $[z^r]F(z) \sim e^{-1}$ as $r \to \infty$.
- Since there are no more poles, we can push C to ∞ in this case, so the error in the approximation decays faster than any exponential.

"Singularity analysis" (Flajolet-Odlyzko 1990)

- Assume F is analytic in a "Camembert" region.

"Singularity analysis" (Flajolet-Odlyzko 1990)

- Assume F is analytic in a "Camembert" region.
- Choose an appropriate ("Hankel") contour approaching the singularity at distance $1/r$.

"Singularity analysis" (Flajolet-Odlyzko 1990)

- Assume F is analytic in a "Camembert" region.
- Choose an appropriate ("Hankel") contour approaching the singularity at distance $1/r$.
- This yields asymptotics for $[z^r]F(z)$ where F looks like $(1-z)^\alpha[-\log(1-z)]^\beta$. "Looks like" means o, O, Θ.

"Singularity analysis" (Flajolet-Odlyzko 1990)

- Assume F is analytic in a "Camembert" region.
- Choose an appropriate ("Hankel") contour approaching the singularity at distance $1/r$.
- This yields asymptotics for $[z^r]F(z)$ where F looks like $(1-z)^\alpha[-\log(1-z)]^\beta$. "Looks like" means o, O, Θ.
- Asymptotics for $F(z)$ near $z=1$ yields asymptotics for $[z^r]F(z)$ automatically. Very useful: singularities in applications are often poles, logarithmic, or square-root.

Darboux' method

- Assume F is of class C^k on $\partial\,\mathrm{U}$. Change variable $z = \rho \exp(i\theta)$, integrate by parts k times. Get

$$a_r = \frac{\rho^{-r}}{2\pi(ir)^k} \int_0^{2\pi} F^{(k)}(e^{i\theta})e^{-ir\theta}.$$

Darboux' method

- Assume F is of class C^k on $\partial\,\mathrm{U}$. Change variable $z = \rho\exp(i\theta)$, integrate by parts k times. Get

$$a_r = \frac{\rho^{-r}}{2\pi(ir)^k}\int_0^{2\pi} F^{(k)}(e^{i\theta})e^{-ir\theta}.$$

- Analyze the oscillating integral using Fourier techniques (Riemann-Lebesgue lemma).

Darboux' method

- Assume F is of class C^k on $\partial\,\mathrm{U}$. Change variable $z = \rho\exp(i\theta)$, integrate by parts k times. Get

$$a_r = \frac{\rho^{-r}}{2\pi(ir)^k} \int_0^{2\pi} F^{(k)}(e^{i\theta})e^{-ir\theta}.$$

- Analyze the oscillating integral using Fourier techniques (Riemann-Lebesgue lemma).
- Can't be used for poles or if F has infinitely many singularities on $\partial\,\mathrm{U}$. In that case, sometimes the circle method of analytic number theory works.

Saddle point method

- Used for "large" (essential) singularities.

Saddle point method

- Used for "large" (essential) singularities.
- Example: $F(z) = \exp(z)$. Cauchy integral formula on a circle C_R of radius R gives $a_n \leq (2\pi)^{-1} F(R)/R^n$.

Saddle point method

- Used for "large" (essential) singularities.
- Example: $F(z) = \exp(z)$. Cauchy integral formula on a circle C_R of radius R gives $a_n \leq (2\pi)^{-1} F(R)/R^n$.
- Consider the "height function" $\log F(R) - n \log R$ and try to minimize over R. In this example, $R = n$ is the minimum.

Saddle point method

- Used for "large" (essential) singularities.
- Example: $F(z) = \exp(z)$. Cauchy integral formula on a circle C_R of radius R gives $a_n \leq (2\pi)^{-1} F(R)/R^n$.
- Consider the "height function" $\log F(R) - n \log R$ and try to minimize over R. In this example, $R = n$ is the minimum.
- We find that the integral over C_R has most mass near $z = n$, so that

$$
\begin{aligned}
a_n &= \frac{1}{2\pi n^n} \int_0^{2\pi} \exp(-in\theta) F(ne^{i\theta}) \, d\theta \\
&\approx \frac{e^n}{2\pi n^n} \int_{-\varepsilon}^{\varepsilon} \exp(-n\theta^2/2 + O(\theta^3)) \, d\theta.
\end{aligned}
$$

Saddle point method

- Used for "large" (essential) singularities.
- Example: $F(z) = \exp(z)$. Cauchy integral formula on a circle C_R of radius R gives $a_n \leq (2\pi)^{-1} F(R)/R^n$.
- Consider the "height function" $\log F(R) - n \log R$ and try to minimize over R. In this example, $R = n$ is the minimum.
- We find that the integral over C_R has most mass near $z = n$, so that

$$\begin{aligned} a_n &= \frac{1}{2\pi n^n} \int_0^{2\pi} \exp(-in\theta) F(ne^{i\theta}) \, d\theta \\ &\approx \frac{e^n}{2\pi n^n} \int_{-\varepsilon}^{\varepsilon} \exp(-n\theta^2/2 + O(\theta^3)) \, d\theta. \end{aligned}$$

- Now Laplace's method gives asymptotics of the integral; leading term is $\sqrt{2\pi/n}$. This gives the first order Stirling formula.

Multivariate asymptotics — some quotations

- (Bender 1974) "Practically nothing is known about asymptotics for recursions in two variables even when a GF is available. Techniques for obtaining asymptotics from bivariate GFs would be quite useful."

Multivariate asymptotics — some quotations

- (Bender 1974) “Practically nothing is known about asymptotics for recursions in two variables even when a GF is available. Techniques for obtaining asymptotics from bivariate GFs would be quite useful.”
- (Odlyzko 1995) “A major difficulty in estimating the coefficients of mvGFs is that the geometry of the problem is far more difficult. . . . Even rational multivariate functions are not easy to deal with.”

Multivariate asymptotics — some quotations

- (Bender 1974) "Practically nothing is known about asymptotics for recursions in two variables even when a GF is available. Techniques for obtaining asymptotics from bivariate GFs would be quite useful."
- (Odlyzko 1995) "A major difficulty in estimating the coefficients of mvGFs is that the geometry of the problem is far more difficult. . . . Even rational multivariate functions are not easy to deal with."
- (Flajolet/Sedgewick 200x) "Roughly, we regard here a bivariate GF as a collection of univariate GFs . . . ."

The mvGF project

- Robin Pemantle (U. Penn.) and I have a major project on mvGF coefficient extraction.

The mvGF project

- Robin Pemantle (U. Penn.) and I have a major project on mvGF coefficient extraction.
- Thoroughly investigate coefficient extraction for meromorphic $F(\mathbf{z}^*) := F(z_1, \ldots, z_d)$ (pole singularities). Amazingly little is known even about rational F in 2 variables.

The mvGF project

- Robin Pemantle (U. Penn.) and I have a major project on mvGF coefficient extraction.
- Thoroughly investigate coefficient extraction for meromorphic $F(\mathbf{z}^*) := F(z_1, \ldots, z_d)$ (pole singularities). Amazingly little is known even about rational F in 2 variables.
- Goal 1: improve over all previous work in generality, ease of use, symmetry, computational effectiveness, uniformity of asymptotics. Create a theory!

The mvGF project

- Robin Pemantle (U. Penn.) and I have a major project on mvGF coefficient extraction.
- Thoroughly investigate coefficient extraction for meromorphic $F(\mathbf{z}^*) := F(z_1, \ldots, z_d)$ (pole singularities). Amazingly little is known even about rational F in 2 variables.
- Goal 1: improve over all previous work in generality, ease of use, symmetry, computational effectiveness, uniformity of asymptotics. Create a theory!
- Goal 2: establish mvGFs as an area worth studying in its own right, a meeting place for many different areas, a common language.

The mvGF project

- Robin Pemantle (U. Penn.) and I have a major project on mvGF coefficient extraction.
- Thoroughly investigate coefficient extraction for meromorphic $F(\mathbf{z}^*) := F(z_1, \ldots, z_d)$ (pole singularities). Amazingly little is known even about rational F in 2 variables.
- Goal 1: improve over all previous work in generality, ease of use, symmetry, computational effectiveness, uniformity of asymptotics. Create a theory!
- Goal 2: establish mvGFs as an area worth studying in its own right, a meeting place for many different areas, a common language.
- Other workers on the project: Yuliy Baryshnikov, Andrew Bressler, Manuel LLadser, Alexander Raichev, Mark Ward.

Some difficulties when $d > 1$

We have no other tool than the Cauchy integral formula.

- Asymptotics:

Some difficulties when $d > 1$

We have no other tool than the Cauchy integral formula.

- Asymptotics:
  - many more ways for $\mathbf{r}$ to go to infinity;

Some difficulties when $d > 1$

We have no other tool than the Cauchy integral formula.

- Asymptotics:
  - many more ways for $\mathbf{r}$ to go to infinity;
  - asymptotics of multivariate integrals are harder to compute.

Some difficulties when $d > 1$

We have no other tool than the Cauchy integral formula.

- Asymptotics:
  - many more ways for $\mathbf{r}$ to go to infinity;
  - asymptotics of multivariate integrals are harder to compute.
- Algebra: rational functions no longer have a partial fraction decomposition.

Some difficulties when $d > 1$

We have no other tool than the Cauchy integral formula.

- Asymptotics:
  - many more ways for $\mathbf{r}$ to go to infinity;
  - asymptotics of multivariate integrals are harder to compute.
- Algebra: rational functions no longer have a partial fraction decomposition.
- Geometry: the singular variety $\mathcal{V}$ is more complicated.

Some difficulties when $d > 1$

We have no other tool than the Cauchy integral formula.

- Asymptotics:
  - many more ways for $\mathbf{r}$ to go to infinity;
  - asymptotics of multivariate integrals are harder to compute.
- Algebra: rational functions no longer have a partial fraction decomposition.
- Geometry: the singular variety $\mathcal{V}$ is more complicated.
  - it does not consist of isolated points;

Some difficulties when $d > 1$

We have no other tool than the Cauchy integral formula.

- Asymptotics:
  - many more ways for $\mathbf{r}$ to go to infinity;
  - asymptotics of multivariate integrals are harder to compute.
- Algebra: rational functions no longer have a partial fraction decomposition.
- Geometry: the singular variety $\mathcal{V}$ is more complicated.
  - it does not consist of isolated points;
  - real dimension of contour is d, that of $\mathcal{V}$ is $2d - 2$, so less room to avoid each other;

Some difficulties when $d > 1$

We have no other tool than the Cauchy integral formula.

- Asymptotics:
  - many more ways for $\mathbf{r}$ to go to infinity;
  - asymptotics of multivariate integrals are harder to compute.
- Algebra: rational functions no longer have a partial fraction decomposition.
- Geometry: the singular variety $\mathcal{V}$ is more complicated.
  - it does not consist of isolated points;
  - real dimension of contour is d, that of $\mathcal{V}$ is $2d - 2$, so less room to avoid each other;
  - topology of $\mathbb{C}^d \setminus \mathcal{V}$ is much more complicated;

Some difficulties when $d > 1$

We have no other tool than the Cauchy integral formula.

- Asymptotics:
  - many more ways for $\mathbf{r}$ to go to infinity;
  - asymptotics of multivariate integrals are harder to compute.
- Algebra: rational functions no longer have a partial fraction decomposition.
- Geometry: the singular variety $\mathcal{V}$ is more complicated.
  - it does not consist of isolated points;
  - real dimension of contour is d, that of $\mathcal{V}$ is $2d - 2$, so less room to avoid each other;
  - topology of $\mathbb{C}^d \setminus \mathcal{V}$ is much more complicated;
- Analysis: the (Leray) residue formula is much harder to use.

Outline of our approach

- Use Cauchy integral formula in $\mathbb{C}^d$, attempt a residue computation. Restrict to meromorphic $F = G/H$ initially.

Outline of our approach

- Use Cauchy integral formula in $\mathbb{C}^d$, attempt a residue computation. Restrict to meromorphic $F = G/H$ initially.
  - Saddle point approach.

Outline of our approach

- Use Cauchy integral formula in $\mathbb{C}^d$, attempt a residue computation. Restrict to meromorphic $F = G/H$ initially.
  - Saddle point approach.
  - Stratified Morse theory.

Outline of our approach

- Use Cauchy integral formula in $\mathbb{C}^d$, attempt a residue computation. Restrict to meromorphic $F = G/H$ initially.
  - Saddle point approach.
  - Stratified Morse theory.
- If geometry of saddle point is reasonably nice:

Outline of our approach

- Use Cauchy integral formula in $\mathbb{C}^d$, attempt a residue computation. Restrict to meromorphic $F = G/H$ initially.
  - Saddle point approach.
  - Stratified Morse theory.
- If geometry of saddle point is reasonably nice:
  - Deform the contour simply so as to write the integral as a $(d-1)$-dimensional integral of a 1-variable residue. Simplify that residue term somehow.

Outline of our approach

- Use Cauchy integral formula in $\mathbb{C}^d$, attempt a residue computation. Restrict to meromorphic $F = G/H$ initially.
  - Saddle point approach.
  - Stratified Morse theory.
- If geometry of saddle point is reasonably nice:
  - Deform the contour simply so as to write the integral as a $(d-1)$-dimensional integral of a 1-variable residue. Simplify that residue term somehow.
  - Convert the outer integral to a Fourier-Laplace integral.

Outline of our approach

- Use Cauchy integral formula in $\mathbb{C}^d$, attempt a residue computation. Restrict to meromorphic $F = G/H$ initially.
  - Saddle point approach.
  - Stratified Morse theory.
- If geometry of saddle point is reasonably nice:
  - Deform the contour simply so as to write the integral as a $(d-1)$-dimensional integral of a 1-variable residue. Simplify that residue term somehow.
  - Convert the outer integral to a Fourier-Laplace integral.
  - Extract asymptotics from the F-L integral using method of stationary phase or similar.

Outline of our approach

- Use Cauchy integral formula in $\mathbb{C}^d$, attempt a residue computation. Restrict to meromorphic $F = G/H$ initially.
  - Saddle point approach.
  - Stratified Morse theory.
- If geometry of saddle point is reasonably nice:
  - Deform the contour simply so as to write the integral as a $(d-1)$-dimensional integral of a 1-variable residue. Simplify that residue term somehow.
  - Convert the outer integral to a Fourier-Laplace integral.
  - Extract asymptotics from the F-L integral using method of stationary phase or similar.
- Otherwise: try resolution of singularities or other approach.

Outline of results

- Asymptotics in the direction $\bar{\mathbf{r}}$ are determined by the geometry of $\mathcal{V}$ near a (finite) set, $\mathrm{crit}(\bar{\mathbf{r}})$, of critical points.

Outline of results

- Asymptotics in the direction $\bar{\mathbf{r}}$ are determined by the geometry of $\mathcal{V}$ near a (finite) set, $\mathrm{crit}(\bar{\mathbf{r}})$, of critical points.
- For computing asymptotics in direction $\bar{\mathbf{r}}$, we may restrict to a subset $\mathrm{contrib}(\bar{\mathbf{r}}) \subseteq \mathrm{crit}(\bar{\mathbf{r}})$ of contributing points.

Outline of results

- Asymptotics in the direction $\overline{\mathbf{r}}$ are determined by the geometry of $\mathcal{V}$ near a (finite) set, $\mathrm{crit}(\overline{\mathbf{r}})$, of critical points.
- For computing asymptotics in direction $\overline{\mathbf{r}}$, we may restrict to a subset $\mathrm{contrib}(\overline{\mathbf{r}}) \subseteq \mathrm{crit}(\overline{\mathbf{r}})$ of contributing points.
- We can determine crit and $\mathrm{contrib}$ by a combination of algebraic and geometric criteria.

Outline of results

- Asymptotics in the direction $\bar{\mathbf{r}}$ are determined by the geometry of $\mathcal{V}$ near a (finite) set, $\mathrm{crit}(\bar{\mathbf{r}})$, of critical points.
- For computing asymptotics in direction $\bar{\mathbf{r}}$, we may restrict to a subset $\mathrm{contrib}(\bar{\mathbf{r}}) \subseteq \mathrm{crit}(\bar{\mathbf{r}})$ of contributing points.
- We can determine crit and $\mathrm{contrib}$ by a combination of algebraic and geometric criteria.
- For each $\mathbf{z}^* \in \mathrm{contrib}$, there is an asymptotic expansion $\mathrm{formula}(\mathbf{z}^*)$ for $a_{\mathbf{r}}$, computable via derivatives of G and H.

Outline of results

- Asymptotics in the direction $\bar{\mathbf{r}}$ are determined by the geometry of $\mathcal{V}$ near a (finite) set, $\mathrm{crit}(\bar{\mathbf{r}})$, of critical points.
- For computing asymptotics in direction $\bar{\mathbf{r}}$, we may restrict to a subset $\mathrm{contrib}(\bar{\mathbf{r}}) \subseteq \mathrm{crit}(\bar{\mathbf{r}})$ of contributing points.
- We can determine crit and $\mathrm{contrib}$ by a combination of algebraic and geometric criteria.
- For each $\mathbf{z}^* \in \mathrm{contrib}$, there is an asymptotic expansion $\mathrm{formula}(\mathbf{z}^*)$ for $a_{\mathbf{r}}$, computable via derivatives of G and H.
- This yields

$$a_{\mathbf{r}} \sim \sum_{\mathbf{z}^* \in \mathrm{contrib}} \mathrm{formula}(\mathbf{z}^*)$$

  where $\mathrm{formula}(\mathbf{z}^*)$ is an asymptotic series that depends on the type of geometry of $\mathcal{V}$ near $\mathbf{z}^*$, and is uniform on compact subsets provided the geometry does not change.

Generic shape of $\mathrm{formula}(\mathbf{z}^*)$

- (smooth point, or multiple point with $n \leq d$)

$$\mathbf{z}^{*-\mathbf{r}} \sum_k a_k |\mathbf{r}|^{-(d-1)/2-k}.$$

Generic shape of $\mathrm{formula}(\mathbf{z}^*)$

- (smooth point, or multiple point with $n \leq d$)

$$\mathbf{z}^{*-\mathbf{r}} \sum_k a_k |\mathbf{r}|^{-(d-1)/2-k}.$$

  - (smooth/multiple point $n < d$)

$$a_0 = G(\mathbf{z}^*) C(\mathbf{z}^*)$$

  where C depends on the derivatives to order 2 of H;

Generic shape of formula($\mathbf{z}^*$)

- (smooth point, or multiple point with $n \leq d$)

$$\mathbf{z}^{*-\mathbf{r}} \sum_k a_k |\mathbf{r}|^{-(d-1)/2-k}.$$

  - (smooth/multiple point $n < d$)

    $$a_0 = G(\mathbf{z}^*) C(\mathbf{z}^*)$$

    where C depends on the derivatives to order 2 of H;
  - (multiple point, $n = d$)

    $$a_0 = G(\mathbf{z}^*)(\det J)^{-1}$$

    where J is the Jacobian matrix $(\partial H_i / \partial z_j)$, other a_k are zero;

Generic shape of formula($\mathbf{z}^*$)

- (smooth point, or multiple point with $n \leq d$)

$$\mathbf{z}^{*-\mathbf{r}} \sum_k a_k |\mathbf{r}|^{-(d-1)/2-k}.$$

  - (smooth/multiple point $n < d$)

    $$a_0 = G(\mathbf{z}^*)C(\mathbf{z}^*)$$

    where C depends on the derivatives to order 2 of H;
  - (multiple point, $n = d$)

    $$a_0 = G(\mathbf{z}^*)(\det J)^{-1}$$

    where J is the Jacobian matrix $(\partial H_i / \partial z_j)$, other a_k are zero;
- (multiple point, $n \geq d$)

$$\mathbf{z}^{*-\mathbf{r}} G(\mathbf{z}^*) P\left(\frac{r_1}{z_1^*}, \dots, \frac{r_d}{z_d^*}\right),$$

P a piecewise polynomial of degree $n - d$.

Simplest special case in dimension 2

- Suppose that $F = G/H$ has a simple pole at $P = (z^*, w^*)$ and $F(z, w)$ is otherwise analytic for $|z| \leq |z^*|, |w| \leq |w^*|$. Define

$$Q(z, w) = -A^2 B - AB^2 - A^2 z^2 H_{zz} - B^2 w^2 H_{ww} + ABH_{zw}$$

where $A = wH_w, B = zH_z$, all computed at P. Then when $s \to \infty$ with $r/s = B/A$,

$$a_{rs} = (z^*)^{-r}(w^*)^{-s} \left[\frac{G(z^*, w^*)}{\sqrt{2\pi}} \sqrt{\frac{-A}{sQ(z^*, w^*)}} + O(s^{-3/2}) \right].$$

The apparent lack of symmetry is illusory, since $A/s = B/r$.

Simplest special case in dimension 2

- Suppose that $F = G/H$ has a simple pole at $P = (z^*, w^*)$ and $F(z, w)$ is otherwise analytic for $|z| \le |z^*|, |w| \le |w^*|$. Define

$$Q(z,w) = -A^2B - AB^2 - A^2z^2H_{zz} - B^2w^2H_{ww} + ABH_{zw}$$

where $A = wH_w, B = zH_z$, all computed at P. Then when $s \to \infty$ with $r/s = B/A$,

$$a_{rs} = (z^*)^{-r}(w^*)^{-s}\left[\frac{G(z^*,w^*)}{\sqrt{2\pi}}\sqrt{\frac{-A}{sQ(z^*,w^*)}} + O(s^{-3/2})\right].$$

The apparent lack of symmetry is illusory, since $A/s = B/r$.

- This simplest case already covers Pascal, Catalan, Motzkin, Schröder, ... triangles, generalized Dyck paths, ordered forests, sums of IID random variables, Lagrange inversion, ...

Next simplest special case in dimension 2

- Suppose that $F = G/H$ has a pole at $P = (z^*, w^*)$, which is a double point of $\mathcal{V}$, $F(z, w)$ is otherwise analytic for $|z| \leq |z^*|, |w| \leq |w^*|$, and $G(P) \neq 0$. Then as $s \to \infty$ for r/s in a certain cone K,

$$a_{rs} \sim (z^*)^{-r}(w^*)^{-s}\left[\frac{G(z^*, w^*)}{\sqrt{(z^*w^*)^2\,\mathrm{hess}(z^*, w^*)}} + O(e^{-c(r+s)})\right]$$

where hess is the Hessian of H.

Next simplest special case in dimension 2

- Suppose that $F = G/H$ has a pole at $P = (z^*, w^*)$, which is a double point of $\mathcal{V}$, $F(z,w)$ is otherwise analytic for $|z| \leq |z^*|, |w| \leq |w^*|$, and $G(P) \neq 0$. Then as $s \to \infty$ for r/s in a certain cone K,

$$a_{rs} \sim (z^*)^{-r}(w^*)^{-s}\left[\frac{G(z^*,w^*)}{\sqrt{(z^*w^*)^2\,\mathrm{hess}(z^*,w^*)}} + O(e^{-c(r+s)})\right]$$

  where hess is the Hessian of H.
- Note that

Next simplest special case in dimension 2

- Suppose that $F = G/H$ has a pole at $P = (z^*, w^*)$, which is a double point of $\mathcal{V}$, $F(z, w)$ is otherwise analytic for $|z| \leq |z^*|, |w| \leq |w^*|$, and $G(P) \neq 0$. Then as $s \to \infty$ for r/s in a certain cone K,

$$a_{rs} \sim (z^*)^{-r}(w^*)^{-s}\left[\frac{G(z^*, w^*)}{\sqrt{(z^*w^*)^2\,\mathrm{hess}(z^*, w^*)}} + O(e^{-c(r+s)})\right]$$

  where hess is the Hessian of H.
- Note that
  - the expansion holds uniformly over compact subcones of K (defined later);

Next simplest special case in dimension 2

- Suppose that $F = G/H$ has a pole at $P = (z^*, w^*)$, which is a double point of $\mathcal{V}$, $F(z, w)$ is otherwise analytic for $|z| \le |z^*|, |w| \le |w^*|$, and $G(P) \neq 0$. Then as $s \to \infty$ for r/s in a certain cone K,

$$a_{rs} \sim (z^*)^{-r}(w^*)^{-s} \left[\frac{G(z^*, w^*)}{\sqrt{(z^* w^*)^2 \operatorname{hess}(z^*, w^*)}} + O(e^{-c(r+s)}) \right]$$

  where hess is the Hessian of H.
- Note that
  - the expansion holds uniformly over compact subcones of K (defined later);
  - the hypothesis $G(P) \neq 0$ is necessary; when $d > 1$, can have $G(P) = H(P) = 0$ even if G, H are relatively prime.

Example: Delannoy numbers

- Consider walks in $\mathbb{Z}^2$ from $(0,0)$, steps in $(1,0),(0,1),(1,1)$. Here $F(x,y) = (1-x-y-xy)^{-1}$.

Example: Delannoy numbers

- Consider walks in $\mathbb{Z}^2$ from $(0,0)$, steps in $(1,0),(0,1),(1,1)$. Here $F(x,y) = (1-x-y-xy)^{-1}$.
- Note $\mathcal{V}$ is globally smooth and crit turns out to be given by $1-x-y-xy=0, x(1+y)s = y(1+x)r$. There is a unique solution for each r,s.

Example: Delannoy numbers

- Consider walks in $\mathbb{Z}^2$ from $(0,0)$, steps in $(1,0),(0,1),(1,1)$. Here $F(x,y) = (1-x-y-xy)^{-1}$.
- Note $\mathcal{V}$ is globally smooth and crit turns out to be given by $1-x-y-xy=0, x(1+y)s = y(1+x)r$. There is a unique solution for each r, s.
- Solving, and using the smooth point formula above we obtain (uniformly for $r/s, s/r$ away from 0)

$$a_{rs} \sim \left[\frac{\Delta - s}{r}\right]^{-r} \left[\frac{\Delta - r}{s}\right]^{-s} \sqrt{\frac{rs}{2\pi\Delta(r+s-\Delta)^2}}.$$

where $\Delta = \sqrt{r^2+s^2}$.

Example: Delannoy numbers

- Consider walks in $\mathbb{Z}^2$ from $(0,0)$, steps in $(1,0),(0,1),(1,1)$. Here $F(x,y) = (1-x-y-xy)^{-1}$.
- Note $\mathcal{V}$ is globally smooth and crit turns out to be given by $1-x-y-xy=0, x(1+y)s = y(1+x)r$. There is a unique solution for each r,s.
- Solving, and using the smooth point formula above we obtain (uniformly for $r/s, s/r$ away from 0)

$$a_{rs} \sim \left[\frac{\Delta - s}{r}\right]^{-r} \left[\frac{\Delta - r}{s}\right]^{-s} \sqrt{\frac{rs}{2\pi\Delta(r+s-\Delta)^2}}.$$

  where $\Delta = \sqrt{r^2+s^2}$.
- Extracting the diagonal ("central Delannoy numbers") is now easy:

$$a_{rr} \sim (3+2\sqrt{2})^r \frac{1}{4\sqrt{2}(3-2\sqrt{2})} r^{-1/2}.$$

Example: queueing network

- Consider

$$F(x,y) = \frac{1}{(1 - \frac{2x}{3} - \frac{y}{3})(1 - \frac{2y}{3} - \frac{x}{3})}$$

which is the “grand partition function” for a very simple queueing network.

Example: queueing network

- Consider

$$F(x,y) = \frac{1}{(1 - \frac{2x}{3} - \frac{y}{3})(1 - \frac{2y}{3} - \frac{x}{3})}$$

which is the "grand partition function" for a very simple queueing network.

- The point $(1,1)$ is a double point satisfying the above. In the cone $1/2 < r/s < 2$, we have $a_{rs} \sim 3$. Outside, the smooth formula holds.

Obvious questions

- Can we always find asymptotics in a given direction in this way? How do we compute $\mathrm{contrib}$?

Obvious questions

- Can we always find asymptotics in a given direction in this way? How do we compute $\mathrm{contrib}$?
- What about higher order terms in the expansions?

Obvious questions

- Can we always find asymptotics in a given direction in this way? How do we compute $\mathrm{contrib}$?
- What about higher order terms in the expansions?
- How easy is it to compute all these formulae automatically?

Obvious questions

- Can we always find asymptotics in a given direction in this way? How do we compute $\mathrm{contrib}$?
- What about higher order terms in the expansions?
- How easy is it to compute all these formulae automatically?
- How do the asymptotics patch together in various regimes? What happens near the boundary of a cone?

Obvious questions

- Can we always find asymptotics in a given direction in this way? How do we compute $\mathrm{contrib}$?
- What about higher order terms in the expansions?
- How easy is it to compute all these formulae automatically?
- How do the asymptotics patch together in various regimes? What happens near the boundary of a cone?
- How does this method compare with others?

Obvious questions

- Can we always find asymptotics in a given direction in this way? How do we compute $\mathrm{contrib}$?
- What about higher order terms in the expansions?
- How easy is it to compute all these formulae automatically?
- How do the asymptotics patch together in various regimes? What happens near the boundary of a cone?
- How does this method compare with others?
- How does it all work (I want to see the details)?

Book references for this lecture

- E. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, Princeton, 1993.
- L. Hörmander, *The analysis of linear partial differential operators. I.*, Springer, 2003.
- V. Arnol'd, S. Guseĭn-Zade, A. Varchenko, *Singularities of Differentiable Maps*, Birkhaüser 1985, 1988.
- I. Aizenberg and A. Yuzhakov, *Integral representations and residues in multidimensional complex analysis*, American Mathematical Society, 1983.

Notation and basic setup

- We assume $F(\mathbf{z}) = \sum a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}} = G(\mathbf{z})/H(\mathbf{z})$, meromorphic in nontrivial polydisc in $\mathbb{C}^d$. Example: rational functions!

Notation and basic setup

- We assume $F(\mathbf{z}) = \sum a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}} = G(\mathbf{z})/H(\mathbf{z})$, meromorphic in nontrivial polydisc in $\mathbb{C}^d$. Example: rational functions!
- U is the domain of convergence of F; $\mathcal{V} = \{\mathbf{z} |\, H(\mathbf{z}) = 0\}$, the singular variety of F.

Notation and basic setup

- We assume $F(\mathbf{z}) = \sum a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}} = G(\mathbf{z})/H(\mathbf{z})$, meromorphic in nontrivial polydisc in $\mathbb{C}^d$. Example: rational functions!
- U is the domain of convergence of F; $\mathcal{V} = \{\mathbf{z} | \, H(\mathbf{z}) = 0\}$, the singular variety of F.
- Separate $\mathbf{r}$ into direction $\overline{\mathbf{r}} := \mathbf{r}/|\mathbf{r}|$ and amplitude $|\mathbf{r}|$. The set of all admissible directions is denoted Ξ.

Notation and basic setup

- We assume $F(\mathbf{z}) = \sum a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}} = G(\mathbf{z})/H(\mathbf{z})$, meromorphic in nontrivial polydisc in $\mathbb{C}^d$. Example: rational functions!
- U is the domain of convergence of F; $\mathcal{V} = \{\mathbf{z} | \, H(\mathbf{z}) = 0\}$, the singular variety of F.
- Separate $\mathbf{r}$ into direction $\overline{\mathbf{r}} := \mathbf{r}/|\mathbf{r}|$ and amplitude $|\mathbf{r}|$. The set of all admissible directions is denoted $\boldsymbol{\Xi}$.
- $\mathrm{Log}(z_1, \ldots, z_d) := (\log |z_1|, \ldots, \log |z_d|)$. $\mathcal{O}^d$ is the positive orthant in $\mathbb{R}^d \subset \mathbb{C}^d$.

Notation and basic setup

- We assume $F(\mathbf{z}) = \sum a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}} = G(\mathbf{z})/H(\mathbf{z})$, meromorphic in nontrivial polydisc in $\mathbb{C}^d$. Example: rational functions!
- U is the domain of convergence of F; $\mathcal{V} = \{\mathbf{z} | \, H(\mathbf{z}) = 0\}$, the singular variety of F.
- Separate $\mathbf{r}$ into direction $\overline{\mathbf{r}} := \mathbf{r}/|\mathbf{r}|$ and amplitude $|\mathbf{r}|$. The set of all admissible directions is denoted Ξ.
- $\mathrm{Log}(z_1, \ldots, z_d) := (\log|z_1|, \ldots, \log|z_d|)$. $\mathcal{O}^d$ is the positive orthant in $\mathbb{R}^d \subset \mathbb{C}^d$.
- A point of $\mathcal{V}$ can be smooth (manifold), multiple (local intersection of n manifolds) or bad (all other types).

Notation and basic setup

- We assume $F(\mathbf{z}) = \sum a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}} = G(\mathbf{z})/H(\mathbf{z})$, meromorphic in nontrivial polydisc in $\mathbb{C}^d$. Example: rational functions!
- U is the domain of convergence of F; $\mathcal{V} = \{\mathbf{z} | \, H(\mathbf{z}) = 0\}$, the singular variety of F.
- Separate $\mathbf{r}$ into direction $\overline{\mathbf{r}} := \mathbf{r}/|\mathbf{r}|$ and amplitude $|\mathbf{r}|$. The set of all admissible directions is denoted $\boldsymbol{\Xi}$.
- $\mathrm{Log}(z_1, \ldots, z_d) := (\log |z_1|, \ldots, \log |z_d|)$. $\mathcal{O}^d$ is the positive orthant in $\mathbb{R}^d \subset \mathbb{C}^d$.
- A point of $\mathcal{V}$ can be smooth (manifold), multiple (local intersection of n manifolds) or bad (all other types).
- A point of $\mathcal{V}$ is minimal if it lies on $\partial \mathrm{U}$.

Notation and basic setup

- We assume $F(\mathbf{z}) = \sum a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}} = G(\mathbf{z})/H(\mathbf{z})$, meromorphic in nontrivial polydisc in $\mathbb{C}^d$. Example: rational functions!
- U is the domain of convergence of F; $\mathcal{V} = \{\mathbf{z} | \, H(\mathbf{z}) = 0\}$, the singular variety of F.
- Separate $\mathbf{r}$ into direction $\overline{\mathbf{r}} := \mathbf{r}/|\mathbf{r}|$ and amplitude $|\mathbf{r}|$. The set of all admissible directions is denoted $\boldsymbol{\Xi}$.
- $\mathrm{Log}(z_1, \dots, z_d) := (\log |z_1|, \dots, \log |z_d|)$. $\mathcal{O}^d$ is the positive orthant in $\mathbb{R}^d \subset \mathbb{C}^d$.
- A point of $\mathcal{V}$ can be smooth (manifold), multiple (local intersection of n manifolds) or bad (all other types).
- A point of $\mathcal{V}$ is minimal if it lies on $\partial \, \mathrm{U}$.
- $\mathrm{T}(\mathbf{z}^*), \mathrm{D}(\mathbf{z}^*)$ the torus, polydisc centred at $\mathbf{0}$, containing $\mathbf{z}^*$.

Notation and basic setup

- We assume $F(\mathbf{z}) = \sum a_{\mathbf{r}} \mathbf{z}^{\mathbf{r}} = G(\mathbf{z})/H(\mathbf{z})$, meromorphic in nontrivial polydisc in $\mathbb{C}^d$. Example: rational functions!
- U is the domain of convergence of F; $\mathcal{V} = \{\mathbf{z} | \, H(\mathbf{z}) = 0\}$, the singular variety of F.
- Separate $\mathbf{r}$ into direction $\overline{\mathbf{r}} := \mathbf{r}/|\mathbf{r}|$ and amplitude $|\mathbf{r}|$. The set of all admissible directions is denoted $\mathbf{\Xi}$.
- $\mathrm{Log}(z_1, \ldots, z_d) := (\log |z_1|, \ldots, \log |z_d|)$. $\mathcal{O}^d$ is the positive orthant in $\mathbb{R}^d \subset \mathbb{C}^d$.
- A point of $\mathcal{V}$ can be smooth (manifold), multiple (local intersection of n manifolds) or bad (all other types).
- A point of $\mathcal{V}$ is minimal if it lies on $\partial\,\mathrm{U}$.
- $\mathrm{T}(\mathbf{z}^*), \mathrm{D}(\mathbf{z}^*)$ the torus, polydisc centred at $\mathbf{0}$, containing $\mathbf{z}^*$.
- For a smooth point of $\mathcal{V}$, $\mathrm{dir}(\mathbf{z}^*)$ is the direction of $(z_1 H_1, \ldots, z_d H_d)$.

Outline of method

- Use Cauchy integral formula in $\mathbb{C}^d$.

Outline of method

- Use Cauchy integral formula in $\mathbb{C}^d$.
- To each minimal point $\mathbf{z}^* \in \mathcal{V}$ we associate a cone $\mathrm{K}(\mathbf{z}^*)$ of directions.

Outline of method

- Use Cauchy integral formula in $\mathbb{C}^d$.
- To each minimal point $\mathbf{z}^* \in \mathcal{V}$ we associate a cone $\mathrm{K}(\mathbf{z}^*)$ of directions.
- If $\overline{\mathbf{r}}$ is bounded away from $\mathrm{K}(\mathbf{z}^*)$, then $|\mathbf{z}^{*\mathbf{r}} a_{\mathbf{r}}|$ decreases exponentially. We show that if $\overline{\mathbf{r}}$ is in $\mathrm{K}(\mathbf{z}^*)$, then $\mathbf{z}^{*-\mathbf{r}}$ is the right asymptotic order, and develop full asymptotic expansions, on a case-by-case basis.

Outline of method

- Use Cauchy integral formula in $\mathbb{C}^d$.
- To each minimal point $\mathbf{z}^* \in \mathcal{V}$ we associate a cone $\mathrm{K}(\mathbf{z}^*)$ of directions.
- If $\overline{\mathbf{r}}$ is bounded away from $\mathrm{K}(\mathbf{z}^*)$, then $|{\mathbf{z}^*}^{\mathbf{r}} a_{\mathbf{r}}|$ decreases exponentially. We show that if $\overline{\mathbf{r}}$ is in $\mathrm{K}(\mathbf{z}^*)$, then ${\mathbf{z}^*}^{-\mathbf{r}}$ is the right asymptotic order, and develop full asymptotic expansions, on a case-by-case basis.
- So far we have done this by simple contour changes to use 1-variable residue theorem; convert to Fourier-Laplace integral in remaining $d-1$ variables; stationary phase/saddle point analysis of these integrals.

Outline of method

- Use Cauchy integral formula in $\mathbb{C}^d$.
- To each minimal point $\mathbf{z}^* \in \mathcal{V}$ we associate a cone $\mathrm{K}(\mathbf{z}^*)$ of directions.
- If $\overline{\mathbf{r}}$ is bounded away from $\mathrm{K}(\mathbf{z}^*)$, then $|\mathbf{z}^{*\mathbf{r}} a_{\mathbf{r}}|$ decreases exponentially. We show that if $\overline{\mathbf{r}}$ is in $\mathrm{K}(\mathbf{z}^*)$, then $\mathbf{z}^{*-\mathbf{r}}$ is the right asymptotic order, and develop full asymptotic expansions, on a case-by-case basis.
- So far we have done this by simple contour changes to use 1-variable residue theorem; convert to Fourier-Laplace integral in remaining $d-1$ variables; stationary phase/saddle point analysis of these integrals.
- There may be other ways to compute the residue integral; however they are unlikely to be easy: explicit residue computation for $d > 1$ seems difficult.

Cauchy integral formula

- We have

$$a_{\mathbf{r}} = (2\pi i)^{-d} \int_T \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z})\, \mathbf{dz}$$

where $\mathbf{dz} = dz_1 \wedge \cdots \wedge dz_d$ and T is a small torus around the origin.

Cauchy integral formula

- We have
$$a_{\mathbf{r}} = (2\pi i)^{-d} \int_T \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z})\, \mathbf{dz}$$
where $\mathbf{dz} = dz_1 \wedge \cdots \wedge dz_d$ and T is a small torus around the origin.
- We aim to use homotopy/homology to replace T by a contour that is more suitable for explicit computation.

Cauchy integral formula

- We have
$$a_{\mathbf{r}} = (2\pi i)^{-d} \int_T \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z})\, \mathbf{dz}$$
where $\mathbf{dz} = dz_1 \wedge \cdots \wedge dz_d$ and T is a small torus around the origin.
- We aim to use homotopy/homology to replace T by a contour that is more suitable for explicit computation.
- This may involve additional residue terms.

Cauchy integral formula

- We have
$$a_{\mathbf{r}} = (2\pi i)^{-d} \int_T \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z})\, \mathbf{dz}$$
where $\mathbf{dz} = dz_1 \wedge \cdots \wedge dz_d$ and T is a small torus around the origin.
- We aim to use homotopy/homology to replace T by a contour that is more suitable for explicit computation.
- This may involve additional residue terms.
- The homology of $\mathbb{C}^d \setminus \mathcal{V}$ is the key to decomposing the integral.

Cauchy integral formula

- We have
$$a_{\mathbf{r}} = (2\pi i)^{-d} \int_T \mathbf{z}^{-\mathbf{r}-\mathbf{1}} F(\mathbf{z}) \, \mathbf{dz}$$
where $\mathbf{dz} = dz_1 \wedge \cdots \wedge dz_d$ and T is a small torus around the origin.
- We aim to use homotopy/homology to replace T by a contour that is more suitable for explicit computation.
- This may involve additional residue terms.
- The homology of $\mathbb{C}^d \setminus \mathcal{V}$ is the key to decomposing the integral.
- It is natural to try a saddle point/steepest descent approach.

Stratified Morse theory

- Consider $h_{\overline{\mathbf{r}}}(\mathbf{z}) = \overline{\mathbf{r}} \cdot \mathrm{Log}(\mathbf{z})$ as a height function; try to deform contour to minimize $\max h$.

Stratified Morse theory

- Consider $h_{\overline{\mathbf{r}}}(\mathbf{z}) = \overline{\mathbf{r}} \cdot \mathrm{Log}(\mathbf{z})$ as a height function; try to deform contour to minimize $\max h$.
- Critical points of h determine the homology of $\mathbb{C}^d \setminus \mathcal{V}$.

Stratified Morse theory

- Consider $h_{\overline{\mathbf{r}}}(\mathbf{z}) = \overline{\mathbf{r}} \cdot \mathrm{Log}(\mathbf{z})$ as a height function; try to deform contour to minimize $\max h$.
- Critical points of h determine the homology of $\mathbb{C}^d \setminus \mathcal{V}$.
- Variety $\mathcal{V}$ decomposes nicely into finitely many cells, each of which is a complex manifold of dimension $k \leq d-1$. The top dimensional stratum is the set of smooth points.

Stratified Morse theory

- Consider $h_{\overline{\mathbf{r}}}(\mathbf{z}) = \overline{\mathbf{r}} \cdot \operatorname{Log}(\mathbf{z})$ as a height function; try to deform contour to minimize $\max h$.
- Critical points of h determine the homology of $\mathbb{C}^d \setminus \mathcal{V}$.
- Variety $\mathcal{V}$ decomposes nicely into finitely many cells, each of which is a complex manifold of dimension $k \leq d-1$. The top dimensional stratum is the set of smooth points.
- The critical points are those where the restriction of h to a stratum has derivative zero. Generically, there are finite many.

Stratified Morse theory

- Consider $h_{\overline{\mathbf{r}}}(\mathbf{z}) = \overline{\mathbf{r}} \cdot \operatorname{Log}(\mathbf{z})$ as a height function; try to deform contour to minimize $\max h$.
- Critical points of h determine the homology of $\mathbb{C}^d \setminus \mathcal{V}$.
- Variety $\mathcal{V}$ decomposes nicely into finitely many cells, each of which is a complex manifold of dimension $k \le d - 1$. The top dimensional stratum is the set of smooth points.
- The critical points are those where the restriction of h to a stratum has derivative zero. Generically, there are finite many.
- The Cauchy integral decomposes into a sum

$$\sum n_i \int_{C_i} \mathbf{z}^{-\mathbf{r}-\mathbf{1}} \mathbf{F}(\mathbf{z}) \, d\mathbf{z}$$

where C_i is a quasi-local cycle for $\mathbf{z}^{*(i)} \in \operatorname{crit}(\overline{\mathbf{r}})$.

Stratified Morse theory

- Consider $h_{\overline{\mathbf{r}}}(\mathbf{z}) = \overline{\mathbf{r}} \cdot \mathrm{Log}(\mathbf{z})$ as a height function; try to deform contour to minimize $\max h$.
- Critical points of h determine the homology of $\mathbb{C}^d \setminus \mathcal{V}$.
- Variety $\mathcal{V}$ decomposes nicely into finitely many cells, each of which is a complex manifold of dimension $k \leq d-1$. The top dimensional stratum is the set of smooth points.
- The critical points are those where the restriction of h to a stratum has derivative zero. Generically, there are finite many.
- The Cauchy integral decomposes into a sum

$$\sum n_i \int_{C_i} \mathbf{z}^{-\mathbf{r}-\mathbf{1}} \mathbf{F}(\mathbf{z})\, \mathbf{dz}$$

  where C_i is a quasi-local cycle for $\mathbf{z}^{*(i)} \in \mathrm{crit}(\overline{\mathbf{r}})$.
- Key problem: find the highest critical points with nonzero n_i. These form the set $\mathrm{contrib}(\overline{\mathbf{r}})$. Others give exponentially smaller contributions.

Logarithmic domain

- Consider $\log \mathrm{U} = \{\mathbf{x} \in \mathbb{R}^d \mid e^{\mathbf{x}} \in U\}$, the image of $\mathrm{U} \cap \mathcal{O}^d$ under Log.

Logarithmic domain

- Consider $\log \mathrm{U} = \{\mathbf{x} \in \mathbb{R}^d \mid e^{\mathbf{x}} \in U\}$, the image of $\mathrm{U} \cap \mathcal{O}^d$ under Log.
- This is known to be convex with boundary the image of $\partial\, \mathrm{U} \cap \mathcal{O}^d$ under Log.

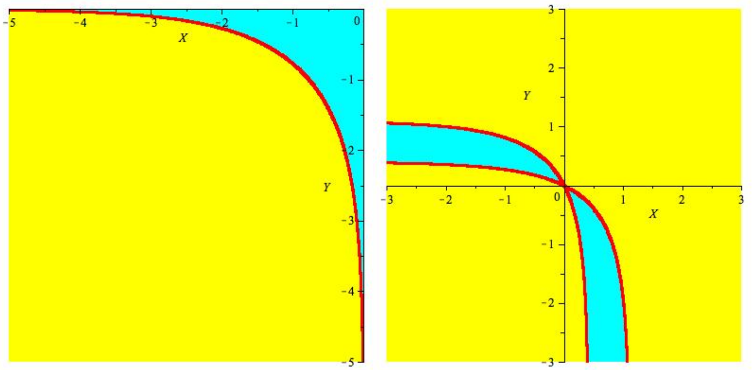
Logarithmic domain

- Consider $\log \mathrm{U} = \{\mathbf{x} \in \mathbb{R}^d \mid e^{\mathbf{x}} \in U\}$, the image of $\mathrm{U} \cap \mathcal{O}^d$ under Log.
- This is known to be convex with boundary the image of $\partial \mathrm{U} \cap \mathcal{O}^d$ under Log.
- (Combinatorial case) Each point of $\partial \log \mathrm{U}$ yields a minimal point of $\mathcal{V}$ that lies in $\mathcal{O}^d$.

Logarithmic domain

- Consider $\log \mathrm{U} = \{\mathbf{x} \in \mathbb{R}^d \mid e^{\mathbf{x}} \in U\}$, the image of $\mathrm{U} \cap \mathcal{O}^d$ under Log.
- This is known to be convex with boundary the image of $\partial \mathrm{U} \cap \mathcal{O}^d$ under Log.
- (Combinatorial case) Each point of $\partial \log \mathrm{U}$ yields a minimal point of $\mathcal{V}$ that lies in $\mathcal{O}^d$.
- The cone spanned by normals to supporting hyperplanes at $\mathbf{x}^* \in \partial \log \mathrm{U}$ we denote by $\mathrm{K}(\mathbf{z}^*)$. If $\mathbf{z}^*$ is smooth, this is a single ray determined by $\mathrm{dir}(\mathbf{z}^*)$, the image of $\mathbf{z}^*$ under the logarithmic Gauss map.

Picture of $\log \mathrm{U}$ for Delannoy and queueing examples



Crit and contrib

- It follows quickly from the definitions that for a smooth point $\mathbf{z}^*$, $\mathbf{z}^* \in \mathrm{crit}(\bar{\mathbf{r}})$ iff $\mathbf{r}$ is a multiple of $\mathrm{dir}(\mathbf{z}^*)$.

Crit and contrib

- It follows quickly from the definitions that for a smooth point $\mathbf{z}^*$, $\mathbf{z}^* \in \operatorname{crit}(\bar{\mathbf{r}})$ iff $\mathbf{r}$ is a multiple of $\operatorname{dir}(\mathbf{z}^*)$.
- Suppose that $\mathbf{z}^*$ is minimal and smooth. Then $\operatorname{dir}(\mathbf{z}^*) \geq \mathbf{0}$ and $\mathbf{z}^* \in \operatorname{contrib}(\bar{\mathbf{r}})$ iff $\mathbf{r}$ is a multiple of $\operatorname{dir}(\mathbf{z}^*)$. All such points must lie on the same torus.

Crit and contrib

- It follows quickly from the definitions that for a smooth point $\mathbf{z}^*$, $\mathbf{z}^* \in \mathrm{crit}(\bar{\mathbf{r}})$ iff $\mathbf{r}$ is a multiple of $\mathrm{dir}(\mathbf{z}^*)$.
- Suppose that $\mathbf{z}^*$ is minimal and smooth. Then $\mathrm{dir}(\mathbf{z}^*) \geq \mathbf{0}$ and $\mathbf{z}^* \in \mathrm{contrib}(\bar{\mathbf{r}})$ iff $\mathbf{r}$ is a multiple of $\mathrm{dir}(\mathbf{z}^*)$. All such points must lie on the same torus.
- These facts extend to multiple points (if $G(\mathbf{z}^*) \neq 0$) by taking the span/convex hull of the dir's of the smooth pieces. But we don't yet know what to do for bad points in general.

Crit and contrib

- It follows quickly from the definitions that for a smooth point $\mathbf{z}^*$, $\mathbf{z}^* \in \mathrm{crit}(\bar{\mathbf{r}})$ iff $\mathbf{r}$ is a multiple of $\mathrm{dir}(\mathbf{z}^*)$.
- Suppose that $\mathbf{z}^*$ is minimal and smooth. Then $\mathrm{dir}(\mathbf{z}^*) \geq \mathbf{0}$ and $\mathbf{z}^* \in \mathrm{contrib}(\bar{\mathbf{r}})$ iff $\mathbf{r}$ is a multiple of $\mathrm{dir}(\mathbf{z}^*)$. All such points must lie on the same torus.
- These facts extend to multiple points (if $G(\mathbf{z}^*) \neq 0$) by taking the span/convex hull of the dir's of the smooth pieces. But we don't yet know what to do for bad points in general.
- Note: for general F, there may not be any minimal points in $\mathrm{contrib}$.

Summary: the aperiodic combinatorial case

- There is an onto map $\bar{\mathbf{r}} \mapsto \mathbf{z}^*$ taking each admissible direction to a minimal point of $\mathcal{V}$ lying in the positive orthant. If all minimal points are smooth, then this map is $1 - 1$.

Summary: the aperiodic combinatorial case

- There is an onto map $\overline{\mathbf{r}} \mapsto \mathbf{z}^*$ taking each admissible direction to a minimal point of $\mathcal{V}$ lying in the positive orthant. If all minimal points are smooth, then this map is $1-1$.
- $\mathbf{z}^*(\overline{\mathbf{r}})$ is the unique element of $\mathrm{contrib}(\overline{\mathbf{r}})$ and is precisely the element of $\mathrm{crit}(\overline{\mathbf{r}})$ that is also a minimal point of $\mathcal{V}$.

Summary: the aperiodic combinatorial case

- There is an onto map $\overline{\mathbf{r}} \mapsto \mathbf{z}^*$ taking each admissible direction to a minimal point of $\mathcal{V}$ lying in the positive orthant. If all minimal points are smooth, then this map is $1-1$.
- $\mathbf{z}^*(\overline{\mathbf{r}})$ is the unique element of $\mathrm{contrib}(\overline{\mathbf{r}})$ and is precisely the element of $\mathrm{crit}(\overline{\mathbf{r}})$ that is also a minimal point of $\mathcal{V}$.
- Thus it suffices to: solve the system $H(\mathbf{z}) = 0, \overline{\mathbf{r}} \in \mathrm{K}(\mathbf{z})$ for $\mathbf{z}^*$; classify local geometry; check for minimality.

Summary: the aperiodic combinatorial case

- There is an onto map $\bar{\mathbf{r}} \mapsto \mathbf{z}^*$ taking each admissible direction to a minimal point of $\mathcal{V}$ lying in the positive orthant. If all minimal points are smooth, then this map is $1 - 1$.
- $\mathbf{z}^*(\bar{\mathbf{r}})$ is the unique element of $\mathrm{contrib}(\bar{\mathbf{r}})$ and is precisely the element of $\mathrm{crit}(\bar{\mathbf{r}})$ that is also a minimal point of $\mathcal{V}$.
- Thus it suffices to: solve the system $H(\mathbf{z}) = 0, \bar{\mathbf{r}} \in \mathrm{K}(\mathbf{z})$ for $\mathbf{z}^*$; classify local geometry; check for minimality.
- All steps but the last are straightforward polynomial algebra for rational F; the last is harder but usually doable.

Summary: the aperiodic combinatorial case

- There is an onto map $\bar{\mathbf{r}} \mapsto \mathbf{z}^*$ taking each admissible direction to a minimal point of $\mathcal{V}$ lying in the positive orthant. If all minimal points are smooth, then this map is $1-1$.
- $\mathbf{z}^*(\bar{\mathbf{r}})$ is the unique element of $\mathrm{contrib}(\bar{\mathbf{r}})$ and is precisely the element of $\mathrm{crit}(\bar{\mathbf{r}})$ that is also a minimal point of $\mathcal{V}$.
- Thus it suffices to: solve the system $H(\mathbf{z}) = 0, \bar{\mathbf{r}} \in \mathrm{K}(\mathbf{z})$ for $\mathbf{z}^*$; classify local geometry; check for minimality.
- All steps but the last are straightforward polynomial algebra for rational F; the last is harder but usually doable.
- We can now use $\mathrm{formula}(\mathbf{z}^*)$ to compute asymptotics in direction $\bar{\mathbf{r}}$. Provided the geometry does not change, the above expansion is uniform (over compact subsets) in $\bar{\mathbf{r}}$.

Sample reduction to iterated integral in simple case

Suppose (WLOG) $(1,1)$ is a smooth or multiple (strictly) minimal point. Here C_a is the circle of radius a centred at 0, $R(z;s;\varepsilon) =$ residue sum in annulus, N a nbhd of 1.

$$
\begin{aligned}
a_{rs} &= (2\pi i)^{-2} \int_{C_1} z^{-r-1} \int_{C_{1-\varepsilon}} w^{-s-1} F(z,w)\, dw\, dz \\
&= (2\pi i)^{-2} \int_N z^{-r-1} \left[\int_{C_{1+\varepsilon}} w^{-s-1} F(z,w) - 2\pi i R(z;s;\varepsilon) \right] dz \\
&\cong -(2\pi i)^{-1} \int_N z^{-r-1} R(z;s;\varepsilon)\, dz \\
&= (2\pi)^{-1} \int_N e^{-ir\theta} (-R(z;s;\varepsilon))\, d\theta.
\end{aligned}
$$

To proceed we need a formula for the residue sum.

Dealing with the residues

- In smooth case, use local parametrization $wv(z) = 1$. Then $R(z; s; \varepsilon) = v(z)^s \operatorname{Res}(F/w)_{|w=1/v(z)} := v(z)^s \psi(z)$. So above has the form

$$(2\pi)^{-1} \int_N \exp\left[-(ir\theta + s \log v(e^{i\theta})\right] (-\psi(e^{i\theta}))\, d\theta.$$

Dealing with the residues

- In smooth case, use local parametrization $wv(z) = 1$. Then $R(z; s; \varepsilon) = v(z)^s \operatorname{Res}(F/w)_{|w=1/v(z)} := v(z)^s \psi(z)$. So above has the form

$$(2\pi)^{-1} \int_N \exp\left[-(ir\theta + s \log v(e^{i\theta})\right] (-\psi(e^{i\theta}))\, d\theta.$$

- In the multiple case there are $n+1$ poles $1/v_0(z), \ldots, 1/v_n(z)$ in the ε-annulus and we use the following nice lemma: Let $h : \mathbb{C} \to \mathbb{C}$ and let μ be the normalized volume measure on the unit simplex $\mathcal{S}_n$. Then

$$\sum_{j=0}^{n} \frac{h(v_j)}{\prod_{r \neq j}(v_j - v_r)} = \int_{\mathcal{S}_n} h^{(n)}(\boldsymbol{\alpha v})\, d\mu(\boldsymbol{\alpha}).$$

Example: Delannoy numbers

- The relevant integral is

$$\int_D \exp\left[ir\theta - s\log\left(\frac{1+z^*e^{i\theta}}{1+z^*}\frac{1-z^*}{1-z^*e^{i\theta}}\right)\right]\frac{1}{1-z^*e^{i\theta}}\,d\theta.$$

Example: Delannoy numbers

- The relevant integral is

$$\int_D \exp\left[ir\theta - s\log\left(\frac{1+z^*e^{i\theta}}{1+z^*}\frac{1-z^*}{1-z^*e^{i\theta}}\right)\right]\frac{1}{1-z^*e^{i\theta}}\,d\theta.$$

- Note that the argument $f(\theta)$ of the exponential has Maclaurin expansion

$$i\left(\frac{r(z^*)^2+2sz^*-r}{(z^*)^2-1}\right)\theta+\frac{sz^*(1+(z^*)^2)}{(1-(z^*)^2)^2}\theta^2+\dots$$

Example: Delannoy numbers

- The relevant integral is

$$\int_D \exp\left[ir\theta - s\log\left(\frac{1+z^*e^{i\theta}}{1+z^*}\frac{1-z^*}{1-z^*e^{i\theta}}\right)\right]\frac{1}{1-z^*e^{i\theta}}\,d\theta.$$

- Note that the argument $f(\theta)$ of the exponential has Maclaurin expansion

$$i\left(\frac{r(z^*)^2+2sz^*-r}{(z^*)^2-1}\right)\theta + \frac{sz^*(1+(z^*)^2)}{(1-(z^*)^2)^2}\theta^2 + \dots$$

- Recall that $\mathrm{crit}(\overline{(r,s)})$ is defined by $1-z-w-zw=0, s(1+w)z = r(1+z)w$. Eliminating w yields $rz^2+2sz-r=0$.

Example: Delannoy numbers

- The relevant integral is

$$\int_D \exp\left[ir\theta - s\log\left(\frac{1+z^*e^{i\theta}}{1+z^*}\frac{1-z^*}{1-z^*e^{i\theta}}\right)\right]\frac{1}{1-z^*e^{i\theta}}\,d\theta.$$

- Note that the argument $f(\theta)$ of the exponential has Maclaurin expansion

$$i\left(\frac{r(z^*)^2+2sz^*-r}{(z^*)^2-1}\right)\theta+\frac{sz^*(1+(z^*)^2)}{(1-(z^*)^2)^2}\theta^2+\dots$$

- Recall that $\mathrm{crit}(\overline{(r,s)})$ is defined by $1-z-w-zw=0, s(1+w)z=r(1+z)w$. Eliminating w yields $rz^2+2sz-r=0$.
- Thus $f(0)=0$, and $f'(0)=0$ because (z^*,w^*) is a critical point for direction $\overline{(r,s)}$.

Fourier-Laplace integrals

- We have been led to large-λ analysis of integrals of the form

$$I(\lambda) = \int_D e^{-\lambda f(\boldsymbol{\theta})} \psi(\boldsymbol{\theta}) \, dV(\boldsymbol{\theta})$$

where:

Fourier-Laplace integrals

- We have been led to large-λ analysis of integrals of the form

$$I(\lambda) = \int_D e^{-\lambda f(\boldsymbol{\theta})} \psi(\boldsymbol{\theta}) \, dV(\boldsymbol{\theta})$$

  where:
  - $\mathbf{0} \in D, f(\mathbf{0}) = 0 = f'(\mathbf{0})$.

Fourier-Laplace integrals

- We have been led to large-λ analysis of integrals of the form

$$I(\lambda) = \int_D e^{-\lambda f(\boldsymbol{\theta})} \psi(\boldsymbol{\theta}) \, dV(\boldsymbol{\theta})$$

where:

  - $\mathbf{0} \in D, f(\mathbf{0}) = 0 = f'(\mathbf{0})$.
  - $\operatorname{Re} f \geq 0$; the phase f and amplitude ψ are analytic.

Fourier-Laplace integrals

- We have been led to large-λ analysis of integrals of the form

$$I(\lambda) = \int_D e^{-\lambda f(\boldsymbol{\theta})} \psi(\boldsymbol{\theta})\, dV(\boldsymbol{\theta})$$

where:

  - $\mathbf{0} \in D, f(\mathbf{0}) = 0 = f'(\mathbf{0})$.
  - $\operatorname{Re} f \geq 0$; the phase f and amplitude ψ are analytic.
  - D is an $(n+d)$-dimensional product of real tori, intervals and simplices; dV the volume element.

Fourier-Laplace integrals

- We have been led to large-λ analysis of integrals of the form

$$I(\lambda) = \int_D e^{-\lambda f(\boldsymbol{\theta})} \psi(\boldsymbol{\theta}) \, dV(\boldsymbol{\theta})$$

  where:
  - $\mathbf{0} \in D, f(\mathbf{0}) = 0 = f'(\mathbf{0})$.
  - $\operatorname{Re} f \geq 0$; the phase f and amplitude ψ are analytic.
  - D is an $(n+d)$-dimensional product of real tori, intervals and simplices; dV the volume element.
- Difficulties in analysis: interplay between exponential and oscillatory decay, nonsmooth boundary of simplex.

Low-dimensional examples of F-L integrals

- Typical smooth point example looks like

$$\int_{-1}^{1} e^{-\lambda(1+i)x^2}\, dx.$$

Isolated nondegenerate critical point, exponential decay

Low-dimensional examples of F-L integrals

- Typical smooth point example looks like

$$\int_{-1}^{1} e^{-\lambda(1+i)x^2}\, dx.$$

  Isolated nondegenerate critical point, exponential decay
- Simplest double point example looks roughly like

$$\int_{-1}^{1}\int_{0}^{1} e^{-\lambda(x^2+2ixy)}\, dy\, dx.$$

  Note $\mathrm{Re}\, f = 0$ on $x = 0$, so rely on oscillation for smallness.

Low-dimensional examples of F-L integrals

- Typical smooth point example looks like

$$\int_{-1}^{1} e^{-\lambda(1+i)x^2}\, dx.$$

Isolated nondegenerate critical point, exponential decay

- Simplest double point example looks roughly like

$$\int_{-1}^{1}\int_{0}^{1} e^{-\lambda(x^2+2ixy)}\, dy\, dx.$$

Note $\operatorname{Re} f = 0$ on $x = 0$, so rely on oscillation for smallness.

- Multiple point with $n = 2, d = 1$ gives integral like

$$\int_{-1}^{1}\int_{0}^{1}\int_{-x}^{x} e^{-\lambda(z^2+2izy)}\, dy\, dx\, dz.$$

Simplex corners now intrude, continuum of critical points.

Asymptotics from F-L integrals

- This is a classical topic with many applications in physics.

Asymptotics from F-L integrals

- This is a classical topic with many applications in physics.
- Exponential decay is much easier to deal with than oscillatory decay.

Asymptotics from F-L integrals

- This is a classical topic with many applications in physics.
- Exponential decay is much easier to deal with than oscillatory decay.
- The main ideas for computing asymptotics of integrals are

Asymptotics from F-L integrals

- This is a classical topic with many applications in physics.
- Exponential decay is much easier to deal with than oscillatory decay.
- The main ideas for computing asymptotics of integrals are
  - (localization) If there is no stationary point of f, then the integral is rapidly decreasing. Thus we can restrict to nbhd of stationary points.

Asymptotics from F-L integrals

- This is a classical topic with many applications in physics.
- Exponential decay is much easier to deal with than oscillatory decay.
- The main ideas for computing asymptotics of integrals are
  - (localization) If there is no stationary point of f, then the integral is rapidly decreasing. Thus we can restrict to nbhd of stationary points.
  - (change of variables/contour moving) ensure that phase has nice form allowing explicit computation of integral.

Asymptotics from F-L integrals

- This is a classical topic with many applications in physics.
- Exponential decay is much easier to deal with than oscillatory decay.
- The main ideas for computing asymptotics of integrals are
  - (localization) If there is no stationary point of f, then the integral is rapidly decreasing. Thus we can restrict to nbhd of stationary points.
  - (change of variables/contour moving) ensure that phase has nice form allowing explicit computation of integral.
  - Integration by parts.

Asymptotics from F-L integrals

- This is a classical topic with many applications in physics.
- Exponential decay is much easier to deal with than oscillatory decay.
- The main ideas for computing asymptotics of integrals are
  - (localization) If there is no stationary point of f, then the integral is rapidly decreasing. Thus we can restrict to nbhd of stationary points.
  - (change of variables/contour moving) ensure that phase has nice form allowing explicit computation of integral.
  - Integration by parts.
- The stationary phase approximation for the leading term, given a quadratically nondegenerate stationary point in the interior of $D \subseteq \mathbb{R}^m$ is

$$\psi(\mathbf{0}) \left(\frac{2\pi}{\lambda}\right)^{m/2} \left(\det f''(\mathbf{0})\right)^{-1/2}.$$

Difficulties with F-L asymptotics

- All authors assume at least one of the following:

Difficulties with F-L asymptotics

- All authors assume at least one of the following:
  - exponential decay on the boundary;

Difficulties with F-L asymptotics

- All authors assume at least one of the following:
  - exponential decay on the boundary;
  - vanishing of amplitude on the boundary;

Difficulties with F-L asymptotics

- All authors assume at least one of the following:
  - exponential decay on the boundary;
  - vanishing of amplitude on the boundary;
  - smooth boundary;

Difficulties with F-L asymptotics

- All authors assume at least one of the following:
  - exponential decay on the boundary;
  - vanishing of amplitude on the boundary;
  - smooth boundary;
  - purely real phase;

Difficulties with F-L asymptotics

- All authors assume at least one of the following:
  - exponential decay on the boundary;
  - vanishing of amplitude on the boundary;
  - smooth boundary;
  - purely real phase;
  - purely imaginary phase;

Difficulties with F-L asymptotics

- All authors assume at least one of the following:
  - exponential decay on the boundary;
  - vanishing of amplitude on the boundary;
  - smooth boundary;
  - purely real phase;
  - purely imaginary phase;
  - isolated stationary point of phase, usually quadratically nondegenerate.

Difficulties with F-L asymptotics

- All authors assume at least one of the following:
  - exponential decay on the boundary;
  - vanishing of amplitude on the boundary;
  - smooth boundary;
  - purely real phase;
  - purely imaginary phase;
  - isolated stationary point of phase, usually quadratically nondegenerate.
- Many of our applications to generating function asymptotics do not fit into this framework. In some cases, we need to extend what is known.

F-L asymptotics: solutions

- If $d = 2$, smooth points, the F-L integral is in dimension 1 and quite sharp results are possible.

F-L asymptotics: solutions

- If $d = 2$, smooth points, the F-L integral is in dimension 1 and quite sharp results are possible.
- For complete intersection and generic smooth points, the F-L integral has a single nondegenerate stationary point in the interior of D, and standard methods apply.

F-L asymptotics: solutions

- If $d = 2$, smooth points, the F-L integral is in dimension 1 and quite sharp results are possible.
- For complete intersection and generic smooth points, the F-L integral has a single nondegenerate stationary point in the interior of D, and standard methods apply.
- For generic multiple points with $n > d$ the F-L integral again has a single nondegenerate stationary point in the interior of D.

F-L asymptotics: solutions

- If $d = 2$, smooth points, the F-L integral is in dimension 1 and quite sharp results are possible.
- For complete intersection and generic smooth points, the F-L integral has a single nondegenerate stationary point in the interior of D, and standard methods apply.
- For generic multiple points with $n > d$ the F-L integral again has a single nondegenerate stationary point in the interior of D.
- For multiple points with $n < d$ we have a higher-dimensional stationary phase set (more difficult).

References for this lecture

- D. Cox, J. Little, D. O'Shea, *Using Algebraic Geometry*, Springer GTM 185, 2005.
- A. Flaxman, A. Harrow, G. Sorkin, EJC 11(1):Research Paper 8, 2004.
- M. Lladser, P. Potočnik, J. Šiagiová, J. Širáň, M. Wilson, *The diameter of random Cayley graphs*, available from my website www.cs.auckland.ac.nz/˜mcw/.
- J. Griggs, P. Hanlon, A. Odlyzko, M. Waterman, *On the number of alignments of k sequences*, Graphs Combin., 6(2):133-146, 1990.

Symbolic computational issues

- If F is rational, points in $\mathrm{crit}(\bar{\mathbf{r}})$ are algebraic over $\mathbb{C}[\mathbf{r}]$. Solving explicitly for crit and $\mathrm{contrib}$ is often not possible. Even when it is, substituting into $\mathrm{formula}$ leads to a huge mess.

Symbolic computational issues

- If F is rational, points in $\mathrm{crit}(\bar{\mathbf{r}})$ are algebraic over $\mathbb{C}[\mathbf{r}]$. Solving explicitly for crit and $\mathrm{contrib}$ is often not possible. Even when it is, substituting into $\mathrm{formula}$ leads to a huge mess.
- It is preferable to work directly with the polynomial ideal I generated by the equations for crit. Gröbner basis and other techniques yield better results.

Symbolic computational issues

- If F is rational, points in $\mathrm{crit}(\bar{\mathbf{r}})$ are algebraic over $\mathbb{C}[\mathbf{r}]$. Solving explicitly for crit and $\mathrm{contrib}$ is often not possible. Even when it is, substituting into $\mathrm{formula}$ leads to a huge mess.
- It is preferable to work directly with the polynomial ideal I generated by the equations for crit. Gröbner basis and other techniques yield better results.
- For example, to simplify Q in the 2-D smooth formula, we may

Symbolic computational issues

- If F is rational, points in $\mathrm{crit}(\bar{\mathbf{r}})$ are algebraic over $\mathbb{C}[\mathbf{r}]$. Solving explicitly for crit and $\mathrm{contrib}$ is often not possible. Even when it is, substituting into $\mathrm{formula}$ leads to a huge mess.
- It is preferable to work directly with the polynomial ideal I generated by the equations for crit. Gröbner basis and other techniques yield better results.
- For example, to simplify Q in the 2-D smooth formula, we may
  - reduce it modulo I;

Symbolic computational issues

- If F is rational, points in $\mathrm{crit}(\bar{\mathbf{r}})$ are algebraic over $\mathbb{C}[\mathbf{r}]$. Solving explicitly for crit and $\mathrm{contrib}$ is often not possible. Even when it is, substituting into $\mathrm{formula}$ leads to a huge mess.
- It is preferable to work directly with the polynomial ideal I generated by the equations for crit. Gröbner basis and other techniques yield better results.
- For example, to simplify Q in the 2-D smooth formula, we may
  - reduce it modulo I;
  - compute its minimal polynomial using a Gröbner basis with an elimination term order;

Symbolic computational issues

- If F is rational, points in $\mathrm{crit}(\bar{\mathbf{r}})$ are algebraic over $\mathbb{C}[\mathbf{r}]$. Solving explicitly for crit and $\mathrm{contrib}$ is often not possible. Even when it is, substituting into $\mathrm{formula}$ leads to a huge mess.
- It is preferable to work directly with the polynomial ideal I generated by the equations for crit. Gröbner basis and other techniques yield better results.
- For example, to simplify Q in the 2-D smooth formula, we may
  - reduce it modulo I;
  - compute its minimal polynomial using a Gröbner basis with an elimination term order;
  - compute its minimal polynomial using the multiplication matrix approach.

Maximum number of distinct subsequences I

- Let a_{nk} be the number of distinct subsequences of length k contained in the prefix of length n of the string $(123\ldots d)^\infty$. Then

$$F(x,y) = \sum_{n,k} a_{nk} x^n y^k = \frac{1}{1 - x - xy(1 - x^d)}.$$

Maximum number of distinct subsequences I

- Let a_{nk} be the number of distinct subsequences of length k contained in the prefix of length n of the string $(123\dots d)^\infty$. Then

$$F(x,y) = \sum_{n,k} a_{nk} x^n y^k = \frac{1}{1-x-xy(1-x^d)}.$$

- Let $\lambda = n/k$. If $\lambda \geq d$, clearly $a_{nk} = d^k$.

Maximum number of distinct subsequences I

- Let a_{nk} be the number of distinct subsequences of length k contained in the prefix of length n of the string $(123\dots d)^\infty$. Then

$$F(x,y) = \sum_{n,k} a_{nk} x^n y^k = \frac{1}{1 - x - xy(1 - x^d)}.$$

- Let $\lambda = n/k$. If $\lambda \geq d$, clearly $a_{nk} = d^k$.
- If $\lambda \leq (d+1)/2$, there is a unique critical point, which is smooth.

Maximum number of distinct subsequences I

- Let a_{nk} be the number of distinct subsequences of length k contained in the prefix of length n of the string $(123\dots d)^\infty$. Then

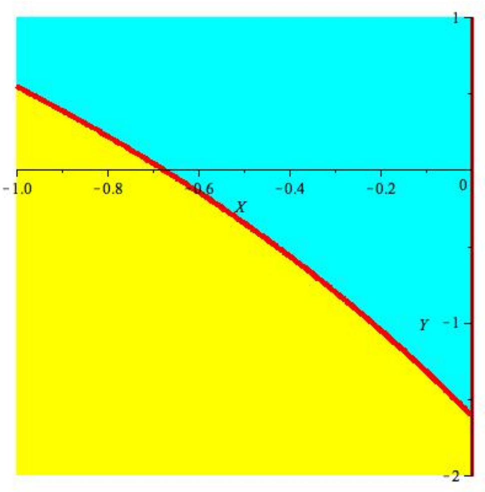
$$F(x,y) = \sum_{n,k} a_{nk} x^n y^k = \frac{1}{1 - x - xy(1 - x^d)}.$$

- Let $\lambda = n/k$. If $\lambda \geq d$, clearly $a_{nk} = d^k$.
- If $\lambda \leq (d+1)/2$, there is a unique critical point, which is smooth.
- If $(d+1)/2 < \lambda < d$, asymptotics are given instead by the double point; get $a_{nk} \sim d^k$.

Maximum number of distinct subsequences I

- Let a_{nk} be the number of distinct subsequences of length k contained in the prefix of length n of the string $(123\ldots d)^\infty$. Then
$$F(x,y) = \sum_{n,k} a_{nk} x^n y^k = \frac{1}{1 - x - xy(1-x^d)}.$$
- Let $\lambda = n/k$. If $\lambda \geq d$, clearly $a_{nk} = d^k$.
- If $\lambda \leq (d+1)/2$, there is a unique critical point, which is smooth.
- If $(d+1)/2 < \lambda < d$, asymptotics are given instead by the double point; get $a_{nk} \sim d^k$.
- Note that
$$F(x,y) = \frac{\phi(x)}{1 - yv(x)}.$$
Above analysis extends to GFs of this form (Riordan arrays).

Maximum number of distinct subsequences: $\log \mathrm{U}$



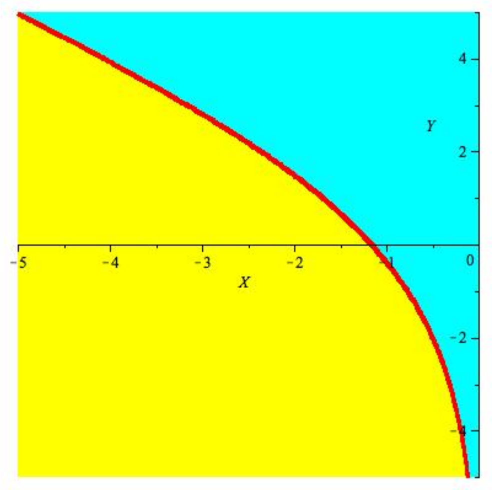
Polyominoes

- The GF for horizontally convex polyominoes ($k =$ rows, $n =$ squares) is

$$F(x, y) = \sum_{n,k} a_{nk} x^n y^k = \frac{xy(1-x)^3}{(1-x)^4 - xy(1 - x - x^2 + x^3 + x^2 y)}.$$

- Generically, $\mathrm{crit}(\bar{\mathbf{r}})$ has 4 points. For each direction with $n/k \geq 1$, there is a contributing point in $\mathcal{O}^2$.
- There are no more (can check that the others are on the wrong torus).

Polyominoes: $\log \mathrm{U}$



Multiple point example — Cayley graph diameters I

- Fix t disjoint pairs from $[n] := \{1, \ldots, n\}$. Let $a(n, k, t)$ be the number of subsets of $[n]$ of size k that do not contain any of the t pairs.

Multiple point example — Cayley graph diameters I

- Fix t disjoint pairs from $[n] := \{1, \ldots, n\}$. Let $a(n, k, t)$ be the number of subsets of $[n]$ of size k that do not contain any of the t pairs.
- The quantity $\binom{n}{k}^{-1} a(n, k, \lfloor (n-1)/12 \rfloor)$ bounds the probability that a random k-valent Cayley digraph on a group of order n has diameter > 2.

Multiple point example — Cayley graph diameters I

- Fix t disjoint pairs from $[n] := \{1, \dots, n\}$. Let $a(n, k, t)$ be the number of subsets of $[n]$ of size k that do not contain any of the t pairs.
- The quantity $\binom{n}{k}^{-1} a(n, k, \lfloor (n-1)/12 \rfloor)$ bounds the probability that a random k-valent Cayley digraph on a group of order n has diameter > 2.
- Relevant GF turns out to be

$$\begin{aligned} F(x, y, z) &= \sum a(n, k, t) x^n y^k z^t \\ &= \left(1 - z(1 - x^2 y^2)\right)^{-1} (1 - x(1 + y))^{-1}. \end{aligned}$$

Multiple point example — Cayley graph diameters I

- Fix t disjoint pairs from $[n] := \{1, \ldots, n\}$. Let $a(n, k, t)$ be the number of subsets of $[n]$ of size k that do not contain any of the t pairs.
- The quantity $\binom{n}{k}^{-1} a(n, k, \lfloor (n-1)/12 \rfloor)$ bounds the probability that a random k-valent Cayley digraph on a group of order n has diameter > 2.
- Relevant GF turns out to be

$$F(x, y, z) = \sum a(n, k, t) x^n y^k z^t$$
$$= \left(1 - z(1 - x^2 y^2)\right)^{-1} (1 - x(1 + y))^{-1}.$$

- Here $a(n, k, t)$ can be negative for large t, so we are not in the combinatorial case. But crit has two elements, both multiple points with $n = 2, d = 3$.

Multiple point example — Cayley graph diameters II

- One point can be eliminated from $\mathrm{contrib}$ since it leads to negative asymptotics for a positive sequence. Answer is asymptotic to

$$C\binom{n}{k}^{-1} x^{-k} y^{-n} z^{-t} n^{-1/2}$$

where x, y, z are quadratic over $\mathbb{Z}[r, s]$.

Multiple point example — Cayley graph diameters II

- One point can be eliminated from contrib since it leads to negative asymptotics for a positive sequence. Answer is asymptotic to

$$C\binom{n}{k}^{-1} x^{-k} y^{-n} z^{-t} n^{-1/2}$$

  where x, y, z are quadratic over $\mathbb{Z}[r, s]$.
- Consider the case $t = \lfloor (n-1)/12 \rfloor$ and linear growth in the generating set $k = \lceil cn \rceil$ for some $c < 1/2$. The exponential growth rate of $\binom{n}{k}^{-1} a(n, k, t)$ is obtained by solving for $\mathrm{crit}(\overline{1, c, 1/12})$. It turns out to be negative, so almost all such digraphs have diameter 2.

Multiple point example — Cayley graph diameters II

- One point can be eliminated from $\mathrm{contrib}$ since it leads to negative asymptotics for a positive sequence. Answer is asymptotic to

$$C\binom{n}{k}^{-1} x^{-k} y^{-n} z^{-t} n^{-1/2}$$

  where x, y, z are quadratic over $\mathbb{Z}[r, s]$.
- Consider the case $t = \lfloor (n-1)/12 \rfloor$ and linear growth in the generating set $k = \lceil cn \rceil$ for some $c < 1/2$. The exponential growth rate of $\binom{n}{k}^{-1} a(n, k, t)$ is obtained by solving for $\mathrm{crit}(\overline{1, c, 1/12})$. It turns out to be negative, so almost all such digraphs have diameter 2.
- More detailed analysis using (parameter-varying) F-L integrals gives results in the sublinear case too.

Alignments example

- A $(d, r_1, \ldots, r_d)$-alignment is a d-row binary matrix with jth row sum r_j and no zero columns.
- The generating function for the number of $(d, \cdot)$-alignments is

$$F(\mathbf{z}) = \sum a(r_1, \ldots, r_d)\mathbf{z}^{\mathbf{r}} = \frac{1}{2 - \prod_{i=1}^{d}(1 + z_i)}.$$

- $\mathcal{V}$ is globally smooth, and we are in the aperiodic combinatorial case. For each $\overline{\mathbf{r}}$, $\operatorname{contrib}(\overline{\mathbf{r}})$ consists of a single element $\mathbf{z}^*(\overline{\mathbf{r}}) \in \mathcal{O}^d$.
- For the diagonal direction we have $\mathbf{z}^*(\overline{\mathbf{1}}) = (2^{1/d} - 1)\mathbf{1}$, so the number of "square" alignments satisfies

$$a(n, n \ldots, n) \sim (2^{1/d} - 1)^{-dn} \frac{1}{(2^{1/d} - 1)2^{(d^2-1)/2d}\sqrt{d(\pi n)^{d-1}}}$$

- Confirms result of [GHOW1990], with less work, and extends to generalized alignments.

Comparing approaches for small singularities

- (GF-sequence methods) Treat $F(z_1, \ldots, z_d)$ as a sequence of $d-1$ dimensional GFs, use probability limit theorems. Pro: can use 1-D methods. Con: complete expansions hard to get, only works well for smooth singularities (below).

Comparing approaches for small singularities

- (GF-sequence methods) Treat $F(z_1, \ldots, z_d)$ as a sequence of $d-1$ dimensional GFs, use probability limit theorems. Pro: can use 1-D methods. Con: complete expansions hard to get, only works well for smooth singularities (below).
- (diagonal method) For each rational slope p/q, consider singularities of $f(t) := F(z^q, t/z^p)$. Pro: gives complete GF for each diagonal using 1-D methods. Con: only works in dimension 2; complexity of computation depends on slope; only rational slopes, so uniform asymptotics impossible.

Comparing approaches for small singularities

- (GF-sequence methods) Treat $F(z_1, \ldots, z_d)$ as a sequence of $d - 1$ dimensional GFs, use probability limit theorems. Pro: can use 1-D methods. Con: complete expansions hard to get, only works well for smooth singularities (below).
- (diagonal method) For each rational slope p/q, consider singularities of $f(t) := F(z^q, t/z^p)$. Pro: gives complete GF for each diagonal using 1-D methods. Con: only works in dimension 2; complexity of computation depends on slope; only rational slopes, so uniform asymptotics impossible.
- (genuinely multivariate methods) Try to use Cauchy residue approach, then convert to Fourier-Laplace integrals. Pro: uniform asymptotics, complete expansions, general approach. Con: geometry of singular set is hard.

Limit laws

- Other authors on multivariate asymptotics concentrate almost exclusively on proving central limit theorems, rather than the "local limit theorems" that we have.

Limit laws

- Other authors on multivariate asymptotics concentrate almost exclusively on proving central limit theorems, rather than the "local limit theorems" that we have.
- CLT holds only in the smooth case where the Hessian is nondegenerate.

Limit laws

- Other authors on multivariate asymptotics concentrate almost exclusively on proving central limit theorems, rather than the "local limit theorems" that we have.
- CLT holds only in the smooth case where the Hessian is nondegenerate.
- Our work also yields a CLT when it applies, but doesn't improve over previous work (nor is it worse). We cover many more general situations too.

Open problems

- Find and classify minimal singularities algorithmically.

Open problems

- Find and classify minimal singularities algorithmically.
- Compute expansions controlled by bad points.

Open problems

- Find and classify minimal singularities algorithmically.
- Compute expansions controlled by bad points.
- Explicit higher order asymptotics from F-L integrals.

Open problems

- Find and classify minimal singularities algorithmically.
- Compute expansions controlled by bad points.
- Explicit higher order asymptotics from F-L integrals.
- Complete analysis of F-L integrals in general case (large stationary phase set).

Open problems

- Find and classify minimal singularities algorithmically.
- Compute expansions controlled by bad points.
- Explicit higher order asymptotics from F-L integrals.
- Complete analysis of F-L integrals in general case (large stationary phase set).
- Patch together asymptotics at cone boundaries; uniformity, phase transitions.

Open problems

- Find and classify minimal singularities algorithmically.
- Compute expansions controlled by bad points.
- Explicit higher order asymptotics from F-L integrals.
- Complete analysis of F-L integrals in general case (large stationary phase set).
- Patch together asymptotics at cone boundaries; uniformity, phase transitions.
- Describe quantities in our formulae geometrically (e.g. using Gauss map).

Work in progress

- (Pemantle, Bressler) Applications to quantum random walks. Here crit is sometimes an entire torus. Treated by a variant of above analysis.
- (Raichev, Wilson) Extending theory to algebraic functions. Currently using reduction of Safonov, which increases dimension by 1, and necessitates higher-order asymptotics.
- (Raichev, Wilson) Explicit higher-order asymptotics for F-L integrals. Applications to algebraic functions and higher moments.
- (Pemantle, Baryshnikov) Derivation of asymptotic formulae controlled by certain bad points (quadratic cones).
- (Lladser, Wilson) Uniform asymptotics near the coordinate planes.