Asymptotics of generalized Riordan arrays

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1. Basic definitions

2. Asymptotics

3. Extensions

Generalized Riordan arrays: their time has come

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- Their asymptotics are, in most cases, routinely derived, yet some researchers still use complicated exact formulae that yield no insight.
- To find out more, read preprint "Twenty combinatorial examples of asymptotics from multivariate generating functions", (soon to be submitted to SIAM Review).

Important background notation

- Multivariate sequence $a : \mathbb{N}^d \to \mathbb{C}$ with multivariate generating function $\sum_{\mathbf{n}} a(\mathbf{n})\mathbf{z}^{\mathbf{n}}$, $\mathbf{z}^{\mathbf{n}} := z_1^{n_1} \cdots z_d^{n_d}$.
- When $d = 2$, we write $F(z, w) = \sum_{n,k} a_{nk} z^n w^k$.
- Radius of convergence of power series f denoted by rad f; order of vanishing at 0 is ord f.

Riordan arrays

- A Riordan array (RA) is an infinite lower triangular complex matrix $M = (a_{nk})_{n\geq 0, k\geq 0}$ having bivariate generating function

$$F(z,w) = \sum_{n,k} a_{nk} z^n w^k = \frac{\phi(z)}{1 - wv(z)}; \qquad \mathrm{ord}(\phi) = 0, \mathrm{ord}(v) \geq 1.$$

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- RAs with $\mathrm{ord}(v) = 1$ (proper RAs) form a group under matrix multiplication. They are heavily used, especially in Firenze, for simplifying combinatorial sums.
- For us it is just as easy to consider generalized RAs (GRAs), where v need not vanish at 0. These correspond to non-triangular matrices.

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- Flaxman/Harrow/Sorkin: maximum number of distinct subsequences.

Relation with lattice walks

- Consider time- and space-homogeneous walks on $\mathbb{Z}^2$, defined by a finite set $E = \{(r_i, s_i) \mid i \in \mathcal{I}\} \subset \mathbb{N} \times \mathbb{Z}$ of jumps.

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    - ★ $E = \{(1,-1),(1,1)\}$ (Dyck paths/ballot numbers)
    - ★ $E = \{(1,-1),(1,0),(1,1)\}$ (Motzkin paths)
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  - $r_i = 1, \max s_i = 1$ (corresponding to walks on $\mathbb{N}$ with steps given by the s_i).
- In fact every nonnegative proper Riordan array arises in this way with $r_i = 1$, provided we allow E to be infinite.

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  - if the apex is $(0, -p)$ then F is algebraic;
  - if the apex has two negative coordinates, F can be non-holonomic.
- Most examples in the literature have apex $(0, 0)$ or $(0, -1)$. This includes all walk examples above, plus everything in Prodinger's "Kernel method: a collection of examples". In this case F is always a GRA.

Equivalent ways of describing the Riordan domain

- generating function of given type;
- exact quasi-power representation, generalized Lagrange inversion;
- triangular arrays with "up and to the right" recurrences;
- directed lattice paths with small positive jumps;
- numbers of nodes in certain generating trees;
- constant coefficient linear recurrences with apex $(0,0)$ or $(0,-1)$; solutions via the kernel method where only one large branch arises.

Multivariate asymptotics background and summary

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- contrib_λ can be computed by algebraic-geometric criteria.
- In particular if $F(z, w) = G(z, w)/H(z, w)$, then asymptotics for $a_{\lambda k,k}$ are controlled by a point solving $zH_z = \lambda w H_w, H = 0$.

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- The aperiodicity constraint can be removed with minor modifications, but nonegativity is essential.

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valid in the direction $\lambda := (zH_z)/(wH_w)$, and uniform as (z, w) varies over a compact set of such points.

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- If P is a double point, then there is a complete asymptotic expansion

$$a_{\lambda k,k} \sim (z^{\lambda} w)^{-k} b_0(\lambda)$$

  uniform in compact subcones of the interior of $\mathsf{K}(P)$. On the boundary, the asymptotic is smaller by a factor of 2.

Simplification of asymptotic formulae in GRA case

- In the smooth case, the leading term is

$$b_0 = \frac{\phi(x)}{\sqrt{2\pi s \sigma^2(v;x)}} \qquad \text{where } \mu(v;x) = \lambda.$$

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- In the double point case (where ϕ has a simple pole), we have

$$b_0(\lambda) = \frac{-\operatorname{Res}(\phi; \rho)}{\rho}.$$

"Explicit" GRA asymptotics: globally smooth case

Theorem

- Let F be an aperiodic nonnegative GRA with $\operatorname{rad}\phi \geq \operatorname{rad} v$. Define $\Delta = [\operatorname{ord} v, \deg v]$. If $\lambda \notin \Delta$ then $a_{\lambda k, k} = 0$.

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- Otherwise there is a unique solution $0 < z_\lambda < \operatorname{rad} v$ to the equation $\mu(v; z) = \lambda$. We have

$$a_{\lambda k,k} \sim [z_\lambda^\lambda v(z_\lambda)]^{-k} k^{-1/2} \sum_{l=0}^{\infty} b_l(\lambda) k^{-l}$$

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  uniformly in λ away from the boundary of Δ.
- The $b_l(\lambda)$ are explicitly computable in terms of derivatives of ϕ and v. The leading coefficient is always

$$b_0(\lambda) = \frac{\phi(z_\lambda)}{\sqrt{2\pi\sigma^2(v; z_\lambda)}}.$$

Examples: lattice paths

Delannoy paths

- Here $v(z) = (1 + z)/(1 - z)$, $\phi(z) = 1/(1 - z)$, so rad ϕ = rad v and above analysis applies.

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- contrib_λ is the minimal positive real solution of $1 - 2z - 3z^2 = \lambda^2$. Thus $z_\lambda = \sqrt{4\lambda^2 - 3}/(3\lambda)$.

Lagrange inversion

- Suppose that $v(z) = zA(v(z))$ with ord $A = 0$. As usual we have for each formal power series ψ

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- Provided $\operatorname{rad} \psi > y_0$, we obtain from above

$$[z^n]\psi(v(z)) \sim A'(y_0)^n n^{-3/2} \sum_{l \geq 0} b_l n^{-l}$$

where

$$b_0 = \frac{y_0 \psi'(y_0)}{\sqrt{2\pi A''(y_0)/A(y_0)}}.$$

"Implicit" GRA asymptotics: globally smooth case

- We can just translate the explicit asymptotics using the Lagrangian form of v.

Theorem

Let (v, ϕ) determine a proper RA, and let $A(y)$ be uniquely defined by $v(z) = zA(v(z))$. If deg $A > 1$ then

$$[z^n]v(z)^k \sim v^{k-n} A^n \frac{k\phi(v/A(v))}{\sqrt{2\pi n^3 \sigma^2(A; v)}} \qquad \text{where } \mu(A; v) = 1 - k/n.$$

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Resymmetrizing: Delannoy paths continued

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  uniformly for every a, b such that $0 < a \le n/k \le b < \infty$.
- The resymmetrizing performed above is not yet automated.

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Asymptotics for subgroups of the Riordan group

Subgroup	Explicit: $\mu(v; x) = n/k$	Implicit: $\mu(A; y) = 1 - k/n$
Bell	$x^{-n} v^{k+1} \frac{1}{\sqrt{2\pi k \sigma^2(v;x)}}$	$y^{k-n} A^{n+1} \frac{k}{\sqrt{2\pi n^3 \sigma^2(A;y)}}$
Hitting time	$x^{-n} v^{k} \frac{n}{\sqrt{2\pi k^3 \sigma^2(v;x)}}$	$v^{k-n} A^{n} \frac{1}{\sqrt{2\pi n \sigma^2(A;v)}}$
Associated	$x^{-n} v^{k} \frac{1}{\sqrt{2\pi k \sigma^2(v;x)}}$	$v^{k-n} A^{n} \frac{k}{\sqrt{2\pi n^3 \sigma^2(A;v)}}$

GRA asymptotics: modifications in the double point case

- Suppose F is a generalized aperiodic nonnegative Riordan array and $\rho := \operatorname{rad} \phi < \operatorname{rad} v$.
- Here $\Delta = [\operatorname{ord} v, \infty)$. Smooth points yield asymptotics only for an initial subinterval $(\operatorname{ord} v, \beta)$ of directions. The other directions are all given by the double point at $x = \rho, y = 1/\rho$.
- If ρ is a pole of ϕ then our methods apply directly.
- Otherwise we may need to rederive results in each case.

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$$F(z, w) = \sum_{n,k} a_{nk} z^n w^k = \frac{1}{1 - z - zw(1 - z^d)}.$$

This is of Riordan type with $\phi(z) = 1/(1 - z)$ and $v(z) = z + z^2 + \cdots + z^d$. Here $\operatorname{rad} \phi = 1 < \infty = \operatorname{rad} v$ and ϕ has a simple pole at 1.

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- In fact $a_{nk} = d^k$ for $n/k \geq d$, but $a_{nk} < d^k$ for $n/k < d$.

Ideas for further work

- Comparing our variable-k with fixed-k results above, it appears that uniform asymptotics hold generally for $k/n \in [0, \varepsilon]$.
- In the case $\phi = 1$, Drmota has already proved this. We have not yet tried to do so in general. We would use results of Lladser.
- Completely clarify the duality of asymptotics, and prove the Lagrange inversion formula using Riordan group automorphisms.
- Find naturally occurring cases not covered by the above results, and extend the theory to deal with them.

Removing the hypotheses

- If v is periodic, contrib will have more than one point, and cancellation will yield periodic asymptotics. Modifications to the above are routine.
- Strange behaviour can occur if we remove the nonnegativity hypothesis, as exemplified by $v = \phi = 1/(3 - 3z + z^2)$:
  - even in the aperiodic case, there may be more than one contributing point;
  - contributing points need not be on the boundary of the domain of convergence;
  - σ^2 can be zero at a contributing point (Airy phenomena).