Polytopes in social choice

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- Note that all scores are linear expressions in the n_i with constant coefficients. For plurality we have $|a| = n_1 + n_2, |b| = n_3 + n_4, |c| = n_5 + n_6$.
- Question: What is the probability that the election is manipulable by strategic voting, assuming the IAC condition (all voting situations are equally likely)?

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- Let $|a|'$ denote a's score after a strategic attempt as above. Then the attempt is successful if and only if $|b|' > |a|', |c|'$.
- We can express $|a|'$ as a linear combination of the n_i and y, and also eliminate y. This yields $n_i \geq 0$, $\sum_i n_i = n$, and

$$
\begin{aligned}
0 &\leq n_1 + n_2 - n_3 - n_4 \\
0 &\leq n_3 + n_4 - n_5 - n_6 \\
0 &\leq -n_1 - n_2 + n_3 + n_4 + n_6 \\
0 &\leq -n_1 - n_2 + 2n_3 + 2n_4 - n_5 + 2n_2.
\end{aligned}
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- Only very recently have the modern methods become known in the social choice community.

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  - the minimal period of f divides the LCM of denominators of coordinates of vertices of P;
  - the generating function $F(t) = \sum_n f(n)t^n$ (called the Ehrhart series) is rational.

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- Converting between representations can take exponential time.
- There can be exponentially many terms in a naive subdivision.
- Similar problems occur when computing volume, not just lattice point computations.

Modern algorithms for lattice points

- All use a more general representation via rational functions. We consider the sum $F(P;\mathbf{x}) = \sum_{\alpha} \mathbf{x}^{\alpha}$ where α runs over all lattice points in P. Putting $\mathbf{x} = \mathbf{1}$ gives the number of lattice points.

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- The series corresponding to a simple unimodular cone is an easily derived rational function. Thus $F(P;\mathbf{x})$ is a sum of nice rational functions.
- All the denominators are singular at $\mathbf{x} = \mathbf{1}$ and so we use residue theory to evaluate the limit $F(P;\mathbf{1})$.

Software for lattice point counting

- Barvinok's algorithm was later extended to parametrized polytopes. This latter algorithm has been implemented in easily available software `LattE` by Jésus de Loera and coworkers.
- The software gives the Ehrhart series of a polytope presented by linear (in)equalities. From that we can determine $f(n)$ by routine computer algebra once we know the minimal period e.
- The problem of determining e is not known to have a polynomial time algorithm, but this is not an issue in most applications I have seen.
- Other software is available based on similar ideas; this is the best one I have found.

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- This corresponds to a decomposition of P into so-called simple unimodular cones at each vertex. The associated generating function of each cone is rational, and the full one is the sum of these.
- In general there may be exponentially many terms in this decomposition. Barvinok's key idea was that we can subtract

Manipulability of plurality

- Polytope has $e = m = 12$.
- Ehrhart series is given by LattE as

$$\frac{12\,t^{12} + 24\,t^{11} + 44\,t^{10} + 56\,t^{9} + 66\,t^{8} + 64\,t^{7} + 63\,t^{6} + 44\,t^{5} + 30\,t^{4}}{(1-t)^2\,(1-t^3)^4\,(1+t)^4\,(1+t^2)^3}$$

- Routine interpolation gives, for example ($n \equiv 1 \mod 12$)

$$f(n) = \frac{7}{17280}\,n^5 + \frac{1}{108}\,n^4 + \frac{341}{5184}\,n^3 + \frac{5}{36}\,n^2 - \frac{917}{17280}\,n - \frac{209}{1296}$$

- Asymptotic answer under IAC for 3 candidates: $7/24 \approx 0.292$.

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- $F(t) = P(t)/Q(t)$ where $\deg Q = 82, \deg P = 75$. Suspect that $e = 2520$. Finding the quasipolynomial requires computation of 15120 coefficients, plus interpolation.
- Clearly far beyond naive methods, and an open problem until 2006.

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- Polytope has 29 vertices, $m = 12$.

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- Asymptotic answer under IAC for 3 candidates: $1/72$.

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- Simplified version of what happened in Bush vs Gore 2000. Related to Simpson's paradox in statistics.
- For each N can write a relevant polytope. For $N = 7$, polytope has 36 vertices.
- Answer : for example, if $N = 7$ and all voting situations equally likely, we have $9409/46080 \approx 0.20419$.

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- I conjecture that many more applications exist in social sciences of which I am unaware.

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- Serious progress in this area will require researchers in social choice theory to understand in some detail how the algorithms actually work.
- This may even lead to proofs for larger (or general) numbers of candidates when the polytopes concerned have a particularly nice structure.

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