The diameter of random Cayley digraphs

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NEW ZEALAND

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- Cayley graphs are also useful for studying groups: the diameter is the maximum length of words in the generators required to generate G as a semigroup.
- Many combinatorial generation algorithms amount to finding Hamilton cycles in Cayley graphs.

Definitions

- Let G be a finite group and S a set of non-identity elements of G. The Cayley digraph $\Gamma = \mathrm{Cay}(G, S)$ has vertex set G and arcs of the form (g, gs) where $g \in G, s \in S$.

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- By vertex-transitivity of Γ, $\mathrm{diam}(\Gamma) = \max_v \partial(1, v)$.

The main question

- How does $\operatorname{diam}\operatorname{Cay}(G, S)$ behave asymptotically as $n \to \infty$? What relationship between $k := |S|$ and $n := |G|$ must hold in order that the diameter is equal to 2 with high probability?

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- (Upper bound) If $k \geq n/2$ then $\operatorname{diam}\operatorname{Cay}(G, S) = 2$.
- What about the region between $\sqrt{n}$ and $n/2$?

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- As far as we know even the linear case $f(n) = cn, 0 < c < 1/2$ is unexplored. Other interesting special cases: $k = \lfloor n^{\alpha} \rfloor$ for $1/2 < \alpha < 1$.

Overview of results of this section

- For $2t \leq n, k \leq n$ define

$$p(n,k,t) = \binom{n}{k}^{-1} \sum_{i=0}^{t} (-1)^i \binom{t}{i} \binom{n-2i}{k-2i}.$$

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- For elementary abelian 2-groups:

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- We therefore want to know the asymptotics of $p(n,k,t)$ for the given values of t, and for various k depending on n.

A basic estimate

- Let $T(y)$ be the event that there exists a path of length 2 from 1 to y, and let $M = \max_y \Pr \overline{T(y)}$. Then

$$M - \frac{k}{n-1} \leq \Pr(\text{Diam} > 2) \leq (n-1)M.$$

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  - Hence

$$\Pr \overline{T(y)} - \frac{k}{n-1} \leq \Pr(\mathrm{Diam} > 2) \leq \Pr \bigcup_{y \in G^*} \overline{T(y)}.$$

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$$\Pr \bigcap_{x \in I} T(x,y) = \binom{n-1}{k}^{-1} \binom{n-1-2i}{k-2i}.$$
- Suppose we have a set J of t such x's such that the pairs $\{x, x^1 y\}$ are all distinct. Then by inclusion-exclusion
$$\Pr \overline{T(y)} = 1 - \Pr \bigcup_{x \in G^*} T(x,y) \leq 1 - \Pr \bigcup_{x \in J} T(x,y)$$
$$= \binom{n-1}{k}^{-1} \sum_{i=1}^{t} (-1)^{i-1} \binom{t}{i} \binom{n-1-2i}{k-2i}.$$

How big can t be?

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We know that $M \le p(n-1,k,t)$, where $t = |J|$, and we want to maximize t.

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- For general groups, can take $t = \lfloor \frac{n-1-s}{3} \rfloor$ and s is the number of square roots of y in G.
- Fact: no nonidentity element in a finite group has more than $3n/4$ square roots. Thus for general groups we have

$$M \le p(n-1,k,t) \qquad \text{where } t = \lfloor \frac{n-4}{12} \rfloor.$$

Overview of results in this section

- We want asymptotics of $p(n, k, \lfloor \frac{n-4}{12} \rfloor)$. The first step is the exponential rate, namely the asymptotics as $n \to \infty$ of $\mathrm{rate} := n^{-1} \log a(n, k, t)$.

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- For other growth rates of k we derive uniform asymptotics using methods of Manuel Lladser's thesis (reported on in San Miniato).
- Result: if $k = \omega(\sqrt{n \log n})$ then

$$\Pr(\text{Diam}_{n,k} > 2) \to 0 \quad \text{as } n \to \infty.$$

Convergence is exponentially fast if k is linear in n and superpolynomial otherwise.

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- Note that $2t \leq n, k+t \leq n$ in this interpretation.
- The trivariate GF assuming $2t \leq n, k+t \leq n$ is easily derived:

$$\sum_{n,k,t} a(n,k,t) x^n y^k z^t = \frac{1}{1-x(1+y)} \frac{1}{1-zx^2(1+2y)}.$$

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- Analyticity means expansions are uniform in large cones.

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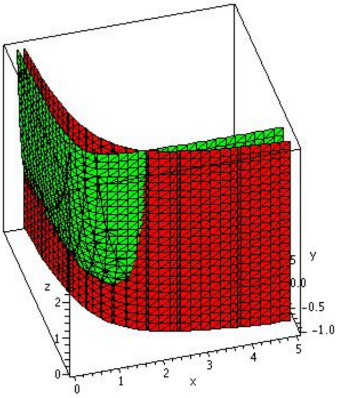
- Upshot: to find asymptotics in direction (n, k, t) we use the contributing point $(1/(1+y), y, (1+y)^2/(1+2y))$ where $y > 0$ and $2(n-k-t)y^2 + (n-3k)y + k = 0$. In the case $k \sim cn, t \sim n/12$, the exponential rate is readily computed.

A picture of the singular variety



Outline of approach in the case of sublinear k

- The contributing points converge to a coordinate axis and the above method requires extension.
- Reduce to a 1-parameter problem: t and k are determined by n, and t is linear in n.
- Use Cauchy's formula in a circle of radius r and convert to a saddle point/stationary phase integral.
- Tune the radius r of the circle of integration in order to capture the correct exponential order.
- Need uniform estimates, obtained by analyticity of the original GF.
- Extract subexponential factors by Laplace's method or similar.

Reduction to a 1-dimensional Fourier-Laplace integral

By expanding the GF, applying Cauchy's integral formula, writing the complex variable in polar form and normalizing we obtain

$$\begin{aligned} a(n,k,t) &= [x^n y^k z^t] \\ &= [y^k](1+y)^{n-2t}(1+2y)^t \\ &= \frac{r^{-k}}{2\pi}\int_{-\pi}^{\pi}(1+re^{i\theta})^{n-2t}(1+2re^{i\theta})^t e^{-ik\theta}\,d\theta \\ &=: (2\pi)^{-1}E(r;n,k,t)I(r;n,k,t) \end{aligned}$$

where

$$E(r;n,k,t) := r^{-k}(1+r)^{n-2t}(1+2r)^t$$

$$I(r;n,k,t) := \int_{-\pi}^{\pi}\left(\frac{1+re^{i\theta}}{1+r}\right)^{n-2t}\left(\frac{1+2re^{i\theta}}{1+2r}\right)^t e^{-ik\theta}\,d\theta.$$

Dealing with $I(r; n, k, t)$

- Write
$$I(r; n, k, t) = \int_{-\pi}^{\pi} e^{-nF(\theta; r, d_1, d_2, d_3)}\, d\theta$$
where
$$F(\theta; r, d_1, d_2, d_3) := id_3\theta - d_1 \log \frac{1 + re^{i\theta}}{1 + r} - d_2 \log \frac{1 + 2re^{i\theta}}{1 + 2r}$$
and $d_1 := (n - 2t)/n, d_2 := t/n, d_3 := k/n$.

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- Upshot: $\operatorname{rate} p(n, k, t) = \operatorname{rate} \binom{n}{k}^{-1} E(r^*; n, k, t)$.

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- Putting it all together with the subexponential factors we obtain the advertised result.

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  - Thus we see that $\Pr_{n,k}(\text{Diam} > 2)$ converges to 0 as $n \to \infty$ provided $k = \omega(\sqrt{n \log n})$.

The threshold

- If $k = \omega(\sqrt{n \log n})$ then $\Pr(\mathrm{Diam}_{n,k} > 2)$ converges to zero and if $k = o(\sqrt{n \log n})$ then our upper bound does not. We conjecture the existence of a sharp phase transition.
- Our lower bound even in the abelian case is too weak to prove this.
- Robin Pemantle has indicated an argument based on Poissonization that confirms the conjecture. We await its appearance!

What next?

- The bounds are fairly crude and general - refine them and specialize for various classes of groups.
- Study the phase transition analytically in much more detail.
- Study the behaviour of Diam when $k \sim c\sqrt{n}$ for c close to 1 (the Moore bound).
- Extend to higher values of diameter?
- Generalize and automate the asymptotic analysis used here in the sublinear case.